ESTIMATING THE PARAMETERS IN MIXTURES OF CIRCULAR AND SPHERICAL DISTRIBUTIONS

Majdi Amine Koutbeiy

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ESTIMATING THE PARAMETERS IN MIXTURES OF CIRCULAR AND SPHERICAL DISTRIBUTIONS

BY

MAJDI AMINE KOUTBEIY

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ST. ANDREWS

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I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate to the Degree of Ph.D.

Signature of Supervisor....   Date...30/8/89...

DECLARATION

I Majdi Amine Koutbeiy hereby certify that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in partial or complete fulfilment of any other degree of professional qualification.

Date 30 – 8 – 89 .

I was admitted to the Faculty of Science of the University of St. Andrews under Ordinance General No 12 in October 1986 and as a candidate for the degree of Ph.D. in October 1987.

signed.....

Date.....30 – 8 – 89.

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Last but not least I would like to thank all my friends, colleagues and members of staff of the Mathematical Institute, University of St. Andrews.

Finally, I wish to dedicate this work to my family.

ABSTRACT

In this thesis we compare various methods for estimating the unknown parameters in mixtures of circular and spherical distributions. We study the von Mises distribution on the circle and the Fisher distribution on the sphere.

We propose a new method of estimation based on the characteristic function and compare it with the classical methods based on maximum likelihood and moments. Thus far these methods have only been successfully applied to distributions on the line. Here we show that the extension to circular and spherical distributions is reasonably straightforward and convergence to the final estimates is fairly rapid. We apply these methods to various simulated and real data sets and show that the results obtained for the mixture of two von Mises distributions are satisfactory but generally depend on the sample size and method of estimation used. However, results obtained for the mixture of two Fisher distributions show that maximum likelihood performs best overall.

CONTENTS

CHAPTER 1

INTRODUCTION

Directional data analysis plays an important part in statistics. In recent years various techniques have been developed to solve the statistical problems which arise in the analysis of directional observations.

Directional data analysis is used in a variety of scientific subjects, for example, in meteorology to analyse wind directions, in geology to interpret palaeomagnetic currents, in astronomy to investigate the origins of comets, and in biology to help in the study of bird migration and navigation. For further details see Mardia(1972), Batschelet(1981) and Fisher, Lewis and Embleton(1987).

In this thesis we deal with directional data in two and three dimensions. For the two dimensional data i.e. observations which are distributed on a circle we study the von Mises distribution. For the three dimensional data i.e. observations which are distributed on a sphere we study the Fisher distribution. These distributions have been extensively studied in the literature, see for example the books by Mardia(1972), Batschelet(1981), Watson(1983) and Fisher, Lewis and Embleton(1987) plus associated papers by various authors. For an up to date bibliography see the recent paper by Jupp and Mardia(1989).

The general problem of estimating the unknown parameters in a mixture of two linear normal distributions is well known and dates back to Karl Pearson(1894). Because many of the early methods involved massive numerical computations which had to be done by hand the problem received little attention in the literature. However, with the advent of modern computers the problem has been extensively studied in recent years and various iterative numerical techniques have been proposed. For an extensive bibliography and further details see Titterington, Smith and Makov(1985).

The analogous problem of estimating the parameters in a mixture of two circular or spherical distributions has received little attention in the literature. Mixtures of circular distributions were discussed by Mardia(1972) and Stephens(1969). They proposed methods based on moments and maximum likelihood respectively. However, they only considered the particular case of the mixture with three unknown parameters i.e. the mixture with equal concentrations and modes directly opposite each other. Spurr(1981) extended these methods to data which are bimodal on the range $(0,\pi)$ and applied them to a geological data set. Mixtures of spherical distributions have rarely been mentioned in the literature although Stephens(1969) and Wood(1982) considered the problem.

In this thesis we consider methods based on (i) maximum likelihood (ii) moments and (iii) minimum distance for estimating the unknown parameters firstly in the mixture of two circular distributions and secondly in the mixture of two spherical distributions. We apply these methods to various simulated data sets plus the well known Turtle data set cited by Stephens(1969) and a spherical data set given by Schmidt(1976). The Turtle data set obtained by Gould on the orientations of Turtles after treatment has been analysed many times in the literature (see Mardia(1972), Mardia (1975a) and Stephens(1969) amongst others). We show that our methods give comparable results to the earlier analyses. The Schmidt data set was analysed by Wood(1982) and Schmidt & McDougall(1977). Again we show our results are in good agreement.

We consider two minimum distance methods ; one based on the Cramer-von Mises distance measure and the other based on a characteristic function transform measure. Also, we provide two approaches to the method of moments ; one based on using five equations constructed by using the first two sine and cosine moments and the derivative of the log likelihood with respect to ρ and the other based on using six equations constructed by using the first three sine and cosine moments.

We find that the methods based on the Cramer-von Mises distance measure and the method of moments based on five equations are very slow to converge and so were not included in all of the comparisons in Section (4.1).

The results obtained in chapter 4 for the mixture of two von Mises distributions indicate that no single method is always better than the others. Sometimes the method of moments based on six equations is the best, sometimes minimum distance based on the characteristic function is the best and sometimes maximum likelihood is the best. However, the CPU times taken when using the method of moments based on six equations and minimum distance based on the characteristic function are similar and considerably quicker than for any of the other methods.

For the spherical case we only consider two methods of estimation - maximum likelihood and minimum distance based on the characteristic function. We considered the method of moments but found difficulties in constructing the seven equations needed and so this method was not pursued.

In fact , optimisation on the sphere is more complicated than on the circle. On the sphere the characteristic function measure no longer reduces to a simpler form as it does on the circle. Consequently, we have to optimise at selected values of the function and this may have an effect on the performance of the method.

The results obtained in chapter 7 for the mixture of two Fisher distributions show that maximum likelihood generally performs better than minimum distance as far as bias and MSE are concerned. However, in most cases, the minimum distance method is quickest with regard to CPU time.

In chapters 2-4 we study the von Mises distribution and in chapters 5-7 we study the Fisher distribution.

Chapter 2 describes the von Mises distribution and also gives a method for simulating the distribution. In Sections 2.1 - 2.4 we study a single von Mises distribution and in Sections 2.5-2.6 a mixture of two von Mises distributions.

Chapter 3 describes the methods of estimation used while in chapter 4 we compare the results for the simulated and real data sets.

Chapter 5 describes the Fisher distribution and its simulation. In Sections 5.1-5.3 we study a single Fisher distribution and in Sections 5.4-5.6 a mixture of two Fisher distributions.

Chapter 6 describes the methods of estimation used and finally chapter 7 compares and discusses the results for the simulated and real data sets.

CHAPTER 2

SIMULATING CIRCULAR DISTRIBUTIONS

2.1 The von Mises Distribution

A circular random variable θ is said to have a von Mises distribution if its probability density function is given by :

$$g(\theta\,;\mu_0\,,\kappa) = \{2\pi\, I_0(\kappa)\}^{-1} \exp\{\kappa \cos(\theta - \mu_0)\} \quad 0 < \theta \le 2\pi\,,\ \kappa > 0\,,\ 0 \le \mu_0 < 2\pi \tag{2.1}$$

where μ_0 is the angle of mean direction , κ is a parameter of concentration and $I_0(\kappa)$ is the modified Bessel function of the first kind and order zero, i.e.

$$I_0(\kappa) = \sum_{r=0}^{\infty} (r!)^{-2} \left\{\frac{\kappa}{2}\right\}^{2r} \tag{2.2}$$

The distribution is unimodal and symmetric about $\theta = \mu_0$; hence , μ_0 is the mode. For a large value of κ the distribution is tightly clustered about $\theta = \mu_0$. For $\kappa = 0$, (2.1) reduces to the uniform distribution. For small values of κ (2.1) reduces to the following model considered by Beran(1969),

$$g(\theta\,;\mu_0\,,\kappa) = \{2\pi\}^{-1} \{\, 1 + \kappa \cos(\theta - \mu_0) \,\} \qquad 0 < \theta \le 2\pi \tag{2.3}$$

This is called the Cardioid distribution. Figure (1) shows the density (2.1) for $\mu_0 = 0^\circ$ and $\kappa = 0.5$, 1 , 5 , 10. Figure (2) shows the density (2.1) for $\kappa = 2$, 10 and $\mu_0 = 0^\circ$, 90° , 180°. Other figures are given in Mardia(1972).

The distribution (2.1) was introduced by von Mises(1918). The von Mises distribution is most frequently used in the analysis of directional data and plays a similar role to that of the normal distribution in linear statistical analysis. Consequently it is often referred to as the Circular Normal Distribution.

Tables of the distribution function are given in Mardia(1972) and Gumbel , Greenwood and Durand(1953). For further details of the distribution see Gumbel et al (1953) and Mardia(1972, pp. 57-64) and for some practical applications see Gumbel(1954).

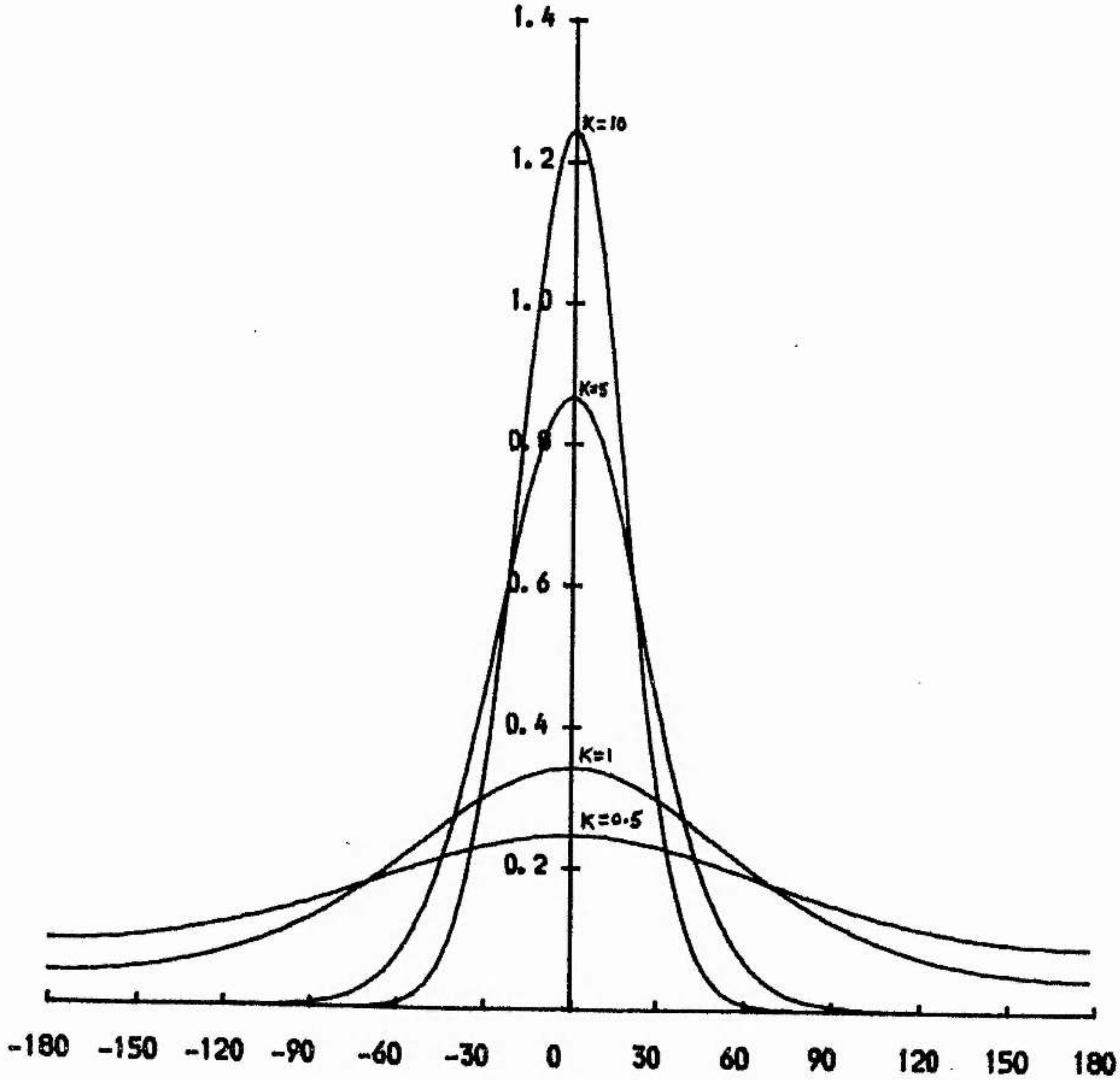
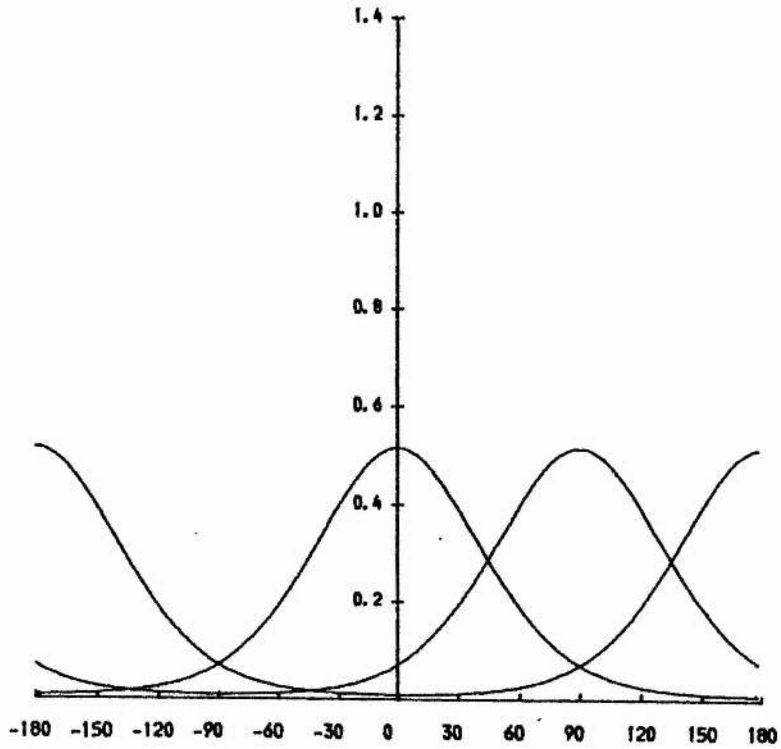
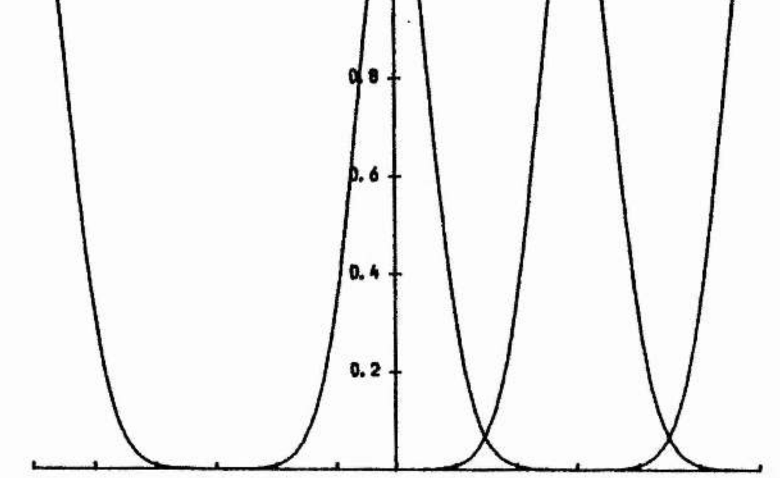


Figure (1)
Density of the von Mises distribution for $\mu_0 = 0$ and $\kappa = 0.5,\ 1,\ 5,\ 10$.

Figure (2)



Density of the von Mises distribution for $\kappa = 2$ and $\mu_0 = 0, 90, 180$.



Density of the von Mises distribution for $\kappa = 10$ and $\mu_0 = 0, 90, 180$.

2.2 Simulation of the von Mises Distribution

We use the algorithm described by Best and Fisher(1979) to simulate samples from the von Mises distribution with mean equal to zero and κ specified. The steps of the algorithm are as follows :

1. Take u_1 , u_2 and u_3 to be pseudo-random observations from U(0,1). New observations u_1 , u_2 and u_3 are used each time steps 5-8 or 10 are executed. Methods for generating these pseudo-random observations are discussed in section 2.3.

2. set $\tau = 1 + (1 + 4 \kappa^2)^{0.5}$,

3. set $\rho = (\tau - (2\tau)^{0.5}) / 2\kappa$,

4. set $r = (1 + \rho^2) / 2\rho$,

5. set $z = \cos(\pi u_1)$,

6. set $f = (1 + r z) / (r + z)$,

7. set $c = \kappa (r - f)$,

8. if $c (2 - c) - u_2 > 0$ go to step 10 .

9. if $\ln (c / u_2) + 1 - c < 0$ return to step 5 .

10. set $\theta = [\text{sign} (u_3 - 0.5)] \cos^{-1}(f)$.

To generate samples with a mean μ not equal to zero we need the following step : -

11. new $\theta = \theta + \mu$.

This algorithm is based on using the wrapped Cauchy distribution and henceforth will be called the wrapped Cauchy method. Best and Fisher compared this method with ones described by Mardia(1972), Siegerstetter(1974) and one using a polynomial envelope. They showed that their method is superior to either Siegerstetter's or the polynomial envelope method , in terms of timings although the latter is reasonable for small κ. They also showed that Mardia's method based on the wrapped normal distribution is good for the extreme cases as $\kappa \to 0$ and $\kappa \to \infty$ but not for intermediate values of κ. Siegerstetter's method also has the disadvantage of being costly to use on the computer. Finally they showed that the wrapped Cauchy method was simple to program and fast for all values of κ and consequently can be considered more efficient than the other methods.

2.3 Random Numbers and Randomness

We have two methods for generating pseudo-random observations from U(0,1). The first method is given by subroutine [G05CAF] of the NAG library and the second method is described in a reference called "Guide for Fortran programming". The second method depends on setting a value for the argument (seed) which is updated automatically. It is recommended that the value of the seed be chosen to be a large odd integer.

We initially chose two different values of the seed namely 99 and 99999. Each value of the seed produces a different set of random observations.

Using the two seeds and the NAG subroutine gave three different ways of generating random observations from U(0,1). In order to choose which method of the three to use , we carried out the following procedure:

1. Using each method, we generated a random sample of size 4000.

2. For each sample we did a Runs test for randomness using the package MINITAB.

3. The results obtained are as follows :

When using the method from the NAG Library we have : -

The observed number of Runs = 2031.

The expected number of Runs = 2000.9995

2001 observations above 0.5 and 1999 below.

The test is significant at α = 0.3429 and so we cannot reject randomness at the 5% level.

When using the second method with seed = 99 we have :

The observed number of Runs = 2000

The expected number of Runs = 2000.0754

1957 observations above 0.5 and 2043 below.

The test is significant at α = 0.9981 and so we cannot reject randomness at the 5% level.

When using the second method with seed = 99999 we have :

The observed number of Runs = 2008

The expected number of Runs = 2000.9955

2003 observations above 0.5 and 1997 below.

The test is significant at α = 0.8247 and so we cannot reject randomness at the 5% level.

Although none of these results indicated non randomness we prefer to use the second method with seed = 99 for generating the random observations from U(0,1), since this gave the least significant result .

2.4 The Chi-squared goodness of fit test for the von Mises distribution

We use the Chi-squared goodness of fit test to test whether the simulated samples adequately fit a von Mises distribution. In order to apply the Chi-squared test we must subdivide the circle into a number of arcs i.e. group the data into suitable angular intervals. In each arc we count the frequency of sample points and calculate the expected frequency from Appendix 2.1 of Mardia(1972) or Table E of Batschelet(1981). The fit of the distribution is considered to be satisfactory if the observed frequencies do not deviate too much from the expected frequencies. The angular intervals need not be equally spaced , but the expected frequency in each interval must not be too small and preferably at least five.

The Chi-squared test statistic is as follows :

$$\chi^2 = \sum_{i=1}^{k} (0_i - e_i)^2 / e_i$$

where k is the number of intervals , 0_i the observed frequency and e_i the expected frequency in the ith group. This statistic is approximately distributed as a χ^2 random variable with k - 1 degrees of freedom.

We now give some examples of testing the fit of simulated data to the von Mises distribution. We test whether the fit is good at a significance level of $\alpha = 0.05$.

	No of observation	The value of κ	No of group intervals	χ^2	5% critical values tables	Conclusion
Example 1	200	0.4	18	12.0774	27.59	accept fit
Example 2	500	0.4	18	21.0.03	27.59	" "
Example 3	200	1.0	12	4.75345	19.68	" "
Example 4	500	1.0	12	11.1378	19.68	" "
Example 5	200	4.0	8	6.94747	14.07	" "
Example 6	500	4.0	9	9.29443	15.51	" "

2.5 Mixtures of von Mises Distributions

The probability density function of a mixture of two von Mises distributions is given by :

$$f(\theta) = p\ f_1(\theta) + (1-p)\ f_2(\theta) \qquad 0 < p < 1\ ,\ 0 < \theta \le 2\pi \tag{2.4}$$

where

$$f_i(\theta) = \{2\pi\, I_0(\kappa_i)\}^{-1} \exp\{\kappa_i \cos(\theta - \mu_i)\} \qquad i = 1,2,\ \kappa_i > 0\ ,\ 0 \le \mu_i < 2\pi\ ,$$

is the von Mises pdf discussed in Section(2.1). p is the mixing parameter. We refer to (2.4) as the five parameter mixture distribution. The special case where $\kappa_1 = \kappa_2$ and $\mu_2 = \mu_1 + \pi$ is referred to as the three parameter mixture distribution.

Model (2.4) was first suggested by Gumbel(1954) in connection with various sets of data, however, he did not use the model because of the analytic difficulties involved.

Jones and James(1969) used maximum likelihood estimation to estimate the unknown parameters in (2.4). They used a combination of two numerical methods to obtain the estimates for a geological data set.

Mardia(1972 pp. 128-130) discussed the analytic difficulties and obtained estimates of the parameters by using the method of moments .

Stephens(1969) also suggested using (2.4) with $\kappa_1 = \kappa_2$ and the two modes 180° apart i.e.

$$f(\theta) = \{2\pi\, I_0(\kappa)\}^{-1} [\, p \exp\{ \kappa \cos(\theta-\mu) \} + (1 - p) \exp\{ -\kappa \cos(\theta-\mu) \} \,] \tag{2.5}$$

although, when fitting to data he used the simpler version of (2.5) with $p = \frac{1}{2}$ i.e.

$$f(\theta) = \{2\pi\, I_0(\kappa)\}^{-1} \cosh(\, \kappa \cos(\theta - \mu)\,) \tag{2.6}$$

This method was extended by Spurr(1981) for bimodal data on the range$(\, 0, \pi\,)$.

May(1967) , Stephens(1969) and Mardia(1972) fitted model (2.5) to the turtle data shown in figure (3). It can be seen from the histogram that this data is bimodally distributed with the modes roughly 180° apart, the two modes being in the intervals $(40^\circ, 80^\circ)$ and $(200^\circ, 280^\circ)$, although they are not of equal strength. They obtained the following best fit values for the parameters :

May estimates are : $p = 0.803$, $\kappa = 3.167$, $\mu = 61.5$,

Stephens estimates are : $p = 0.803$, $\kappa = 3.05$, $\mu = 63.08$ and

Mardia estimates are : $p = 0.78$, $\kappa = 3.553$, $\mu = 61.4$.

In Section (4.2) we fit the general model (2.4) to this data set and discuss the results.

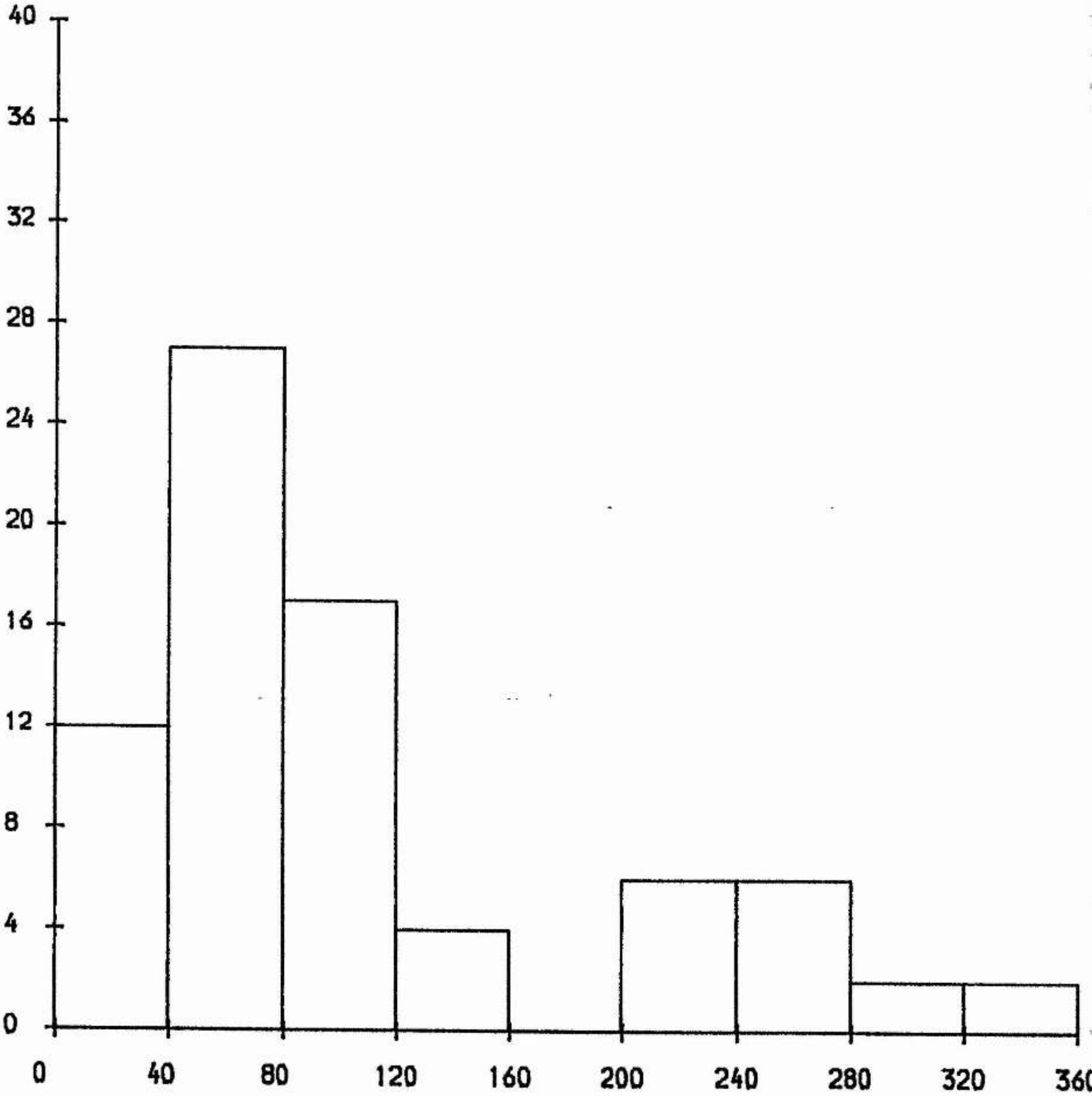


Figure (3)
Histogram plot for the Turtle data cited by Stephens (1969).

2.6 Simulation of Mixtures of von Mises Distributions

To simulate random samples of size n from a mixture of von Mises distributions we do the following : -

1. We simulate n observations from each of $g(\theta ; \mu_1 , \kappa_1)$ and $g(\theta ; \mu_2 , \kappa_2)$ using the previous algorithm of Best & Fisher(1979).

2. We choose the required value of the mixing parameter , p.

3. We generate n random numbers from U(0,1) , say , u1, u2, ...,un.

4. Finally we create a sample of n observations from the specified mixture distribution by applying the following procedure : -

(i) If u1 < p then the first observation from the simulated $g(\theta ; \mu_1 , \kappa_1)$ will be taken into the mixture and if u1 > p then the first observation from the simulated $g(\theta ; \mu_2 , \kappa_2)$ will be taken into the mixture.

(ii) We repeat this process using the remaining random numbers u2 , u3 , ... , un to obtain the required sample from the mixture distribution.

Figures (4) - (9) illustrate the mixture of two von Mises densities for the six examples discussed in Section (4). Also, the figures show the underlying components p f_1 and (1 - p) f_2.

From figures (4) - (7) we see that when the underlying components are well separated the mixture density has two obvious peaks. These two peaks show clearly the two mean directions i.e. μ_1 and μ_2. However, figure (8) clearly shows one peak and a bump where the other peak should be. This happens because the two separate mean directions are very close together although the concentrations are different. Figure (9) clearly shows only one peak because here the two mean directions are very close together and the concentration parameters κ_1 and κ_2 are small and almost equal.

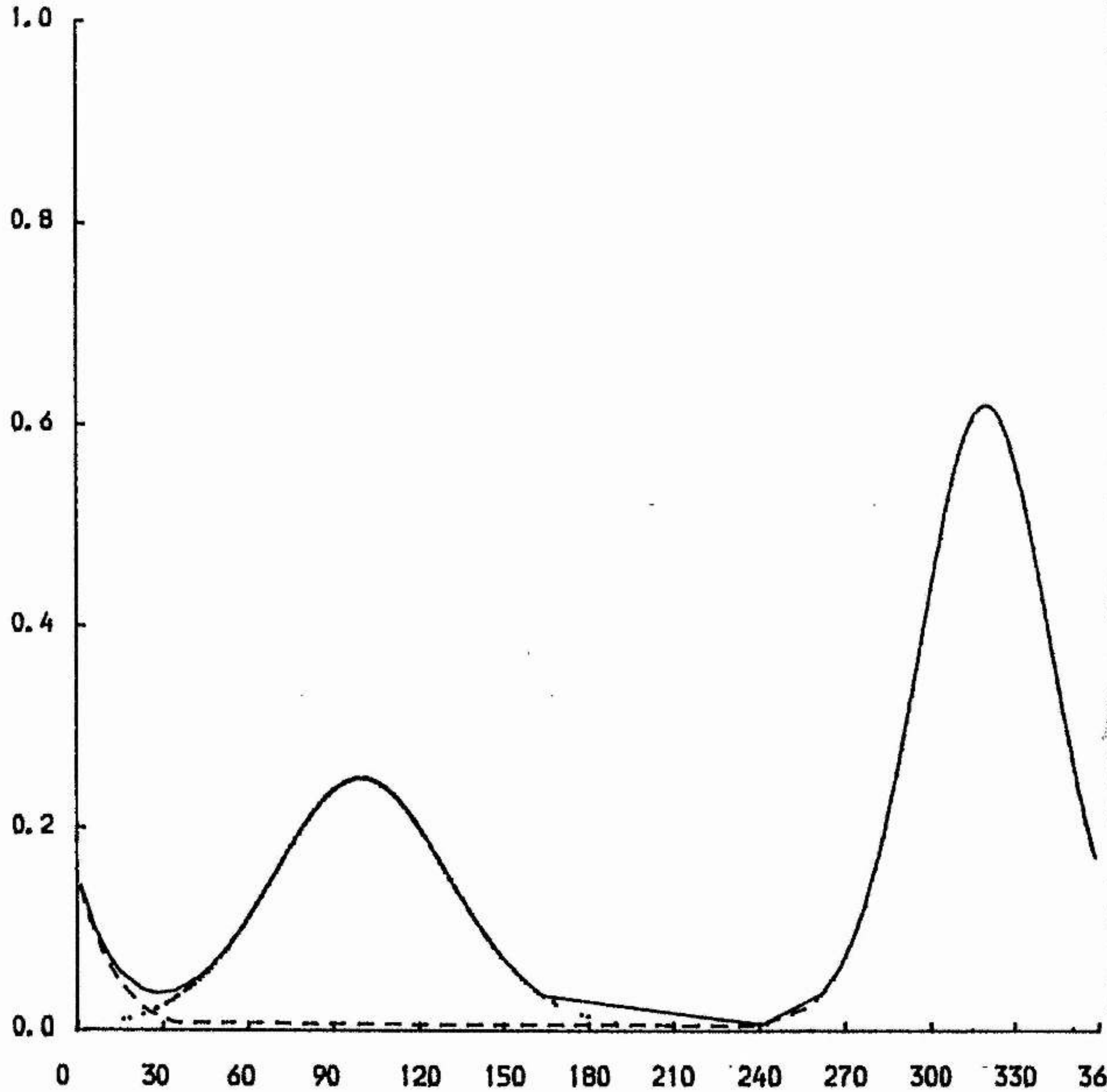


Figure (4)

The mixture density is displayed using the solid curve, and the underlying components $p\ f_1$ and $(1 - p)\ f_2$ are displayed using dotted and dash curves, respectively.

The true values are $p = 0.35$, $\kappa_1 = 3.5$, $\kappa_2 = 6.0$, $\mu_1 = 100.0$, $\mu_2 = 320.0$.

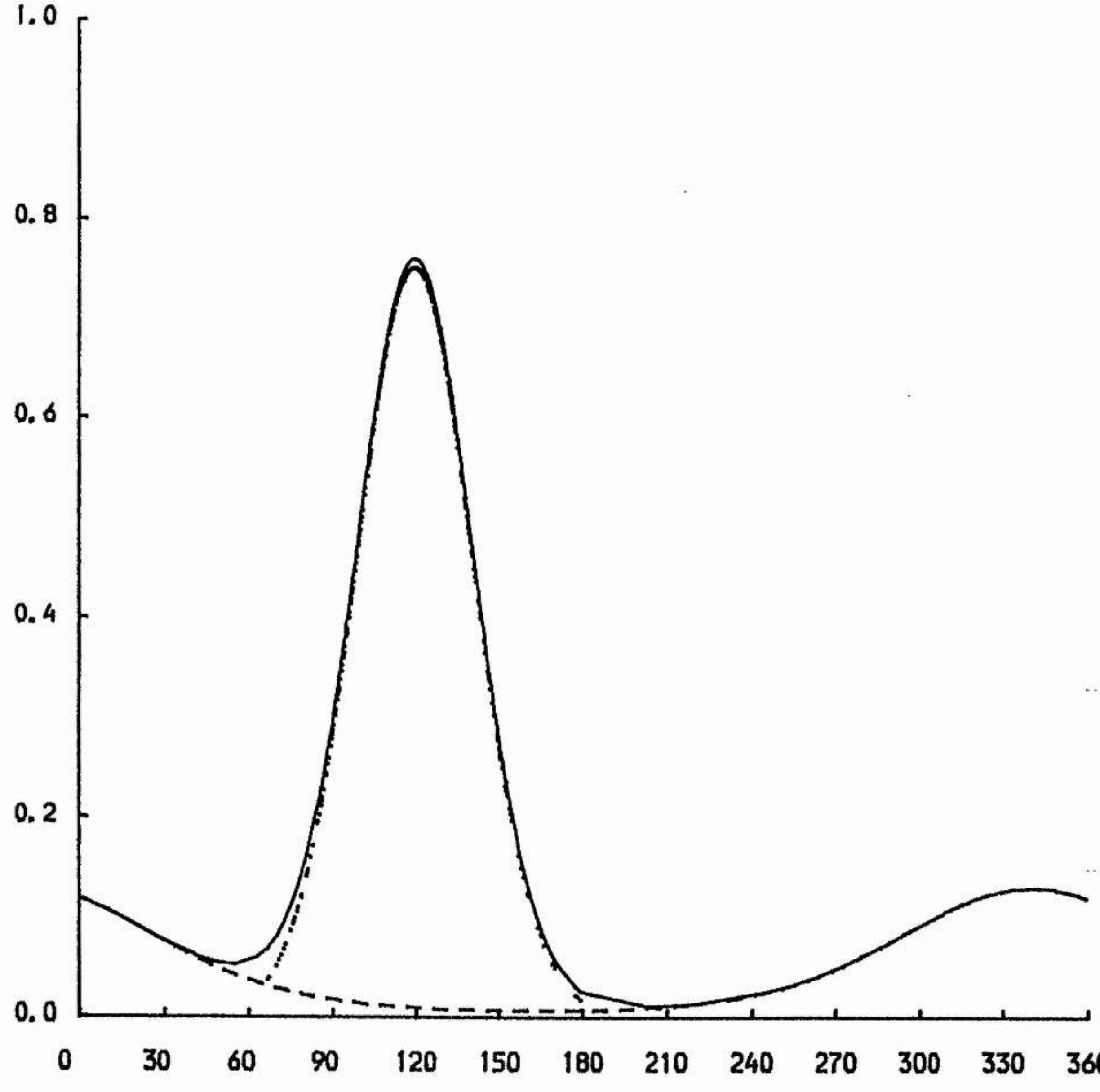


Figure (5)
The mixture density is displayed using the solid curve, and the underlying components p f_1 and (1 - p) f_2 are displayed using dotted and dash curves, respectively.
The true values are $p = 0.70$, $\kappa_1 = 7.5$, $\kappa_2 = 1.5$, $\mu_1 = 120.0$, $\mu_2 = 340.0$.

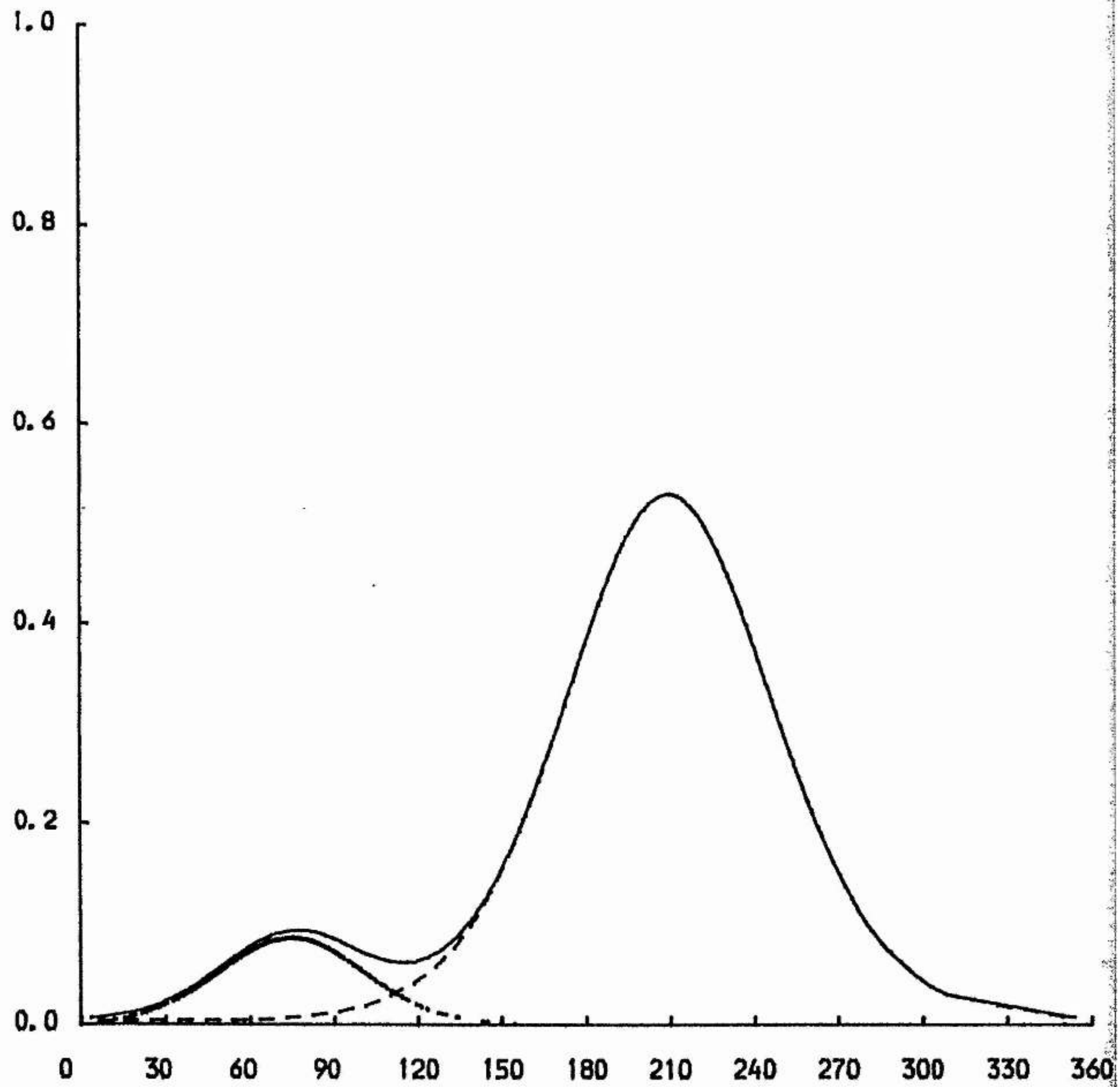


Figure (6)

The mixture density is displayed using the solid curve, and the underlying components $p\ f_1$ and $(1 - p)\ f_2$ are displayed using dotted and dash curves, respectively.

The true values are $p = 0.10$, $\kappa_1 = 5.0$, $\kappa_2 = 2.5$, $\mu_1 = 75.0$, $\mu_2 = 210.0$.

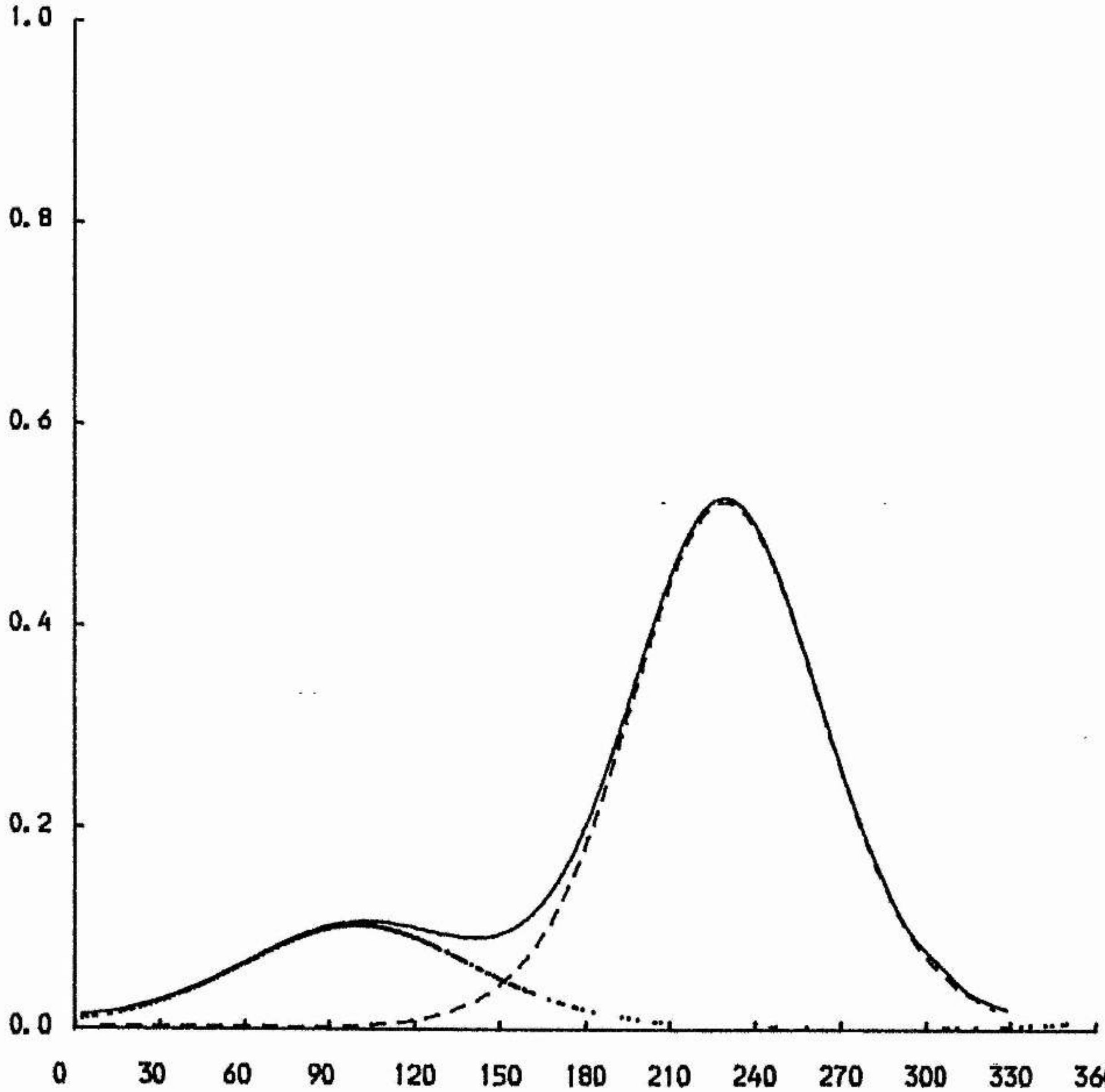


Figure (7)

The mixture density is displayed using the solid curve, and the underlying components $p\ f_1$ and $(1 - p)\ f_2$ are displayed using dotted and dash curves, respectively.

The true values are $p = 0.20$, $\kappa_1 = 2.0$, $\kappa_2 = 3.0$, $\mu_1 = 100.0$, $\mu_2 = 230.0$.

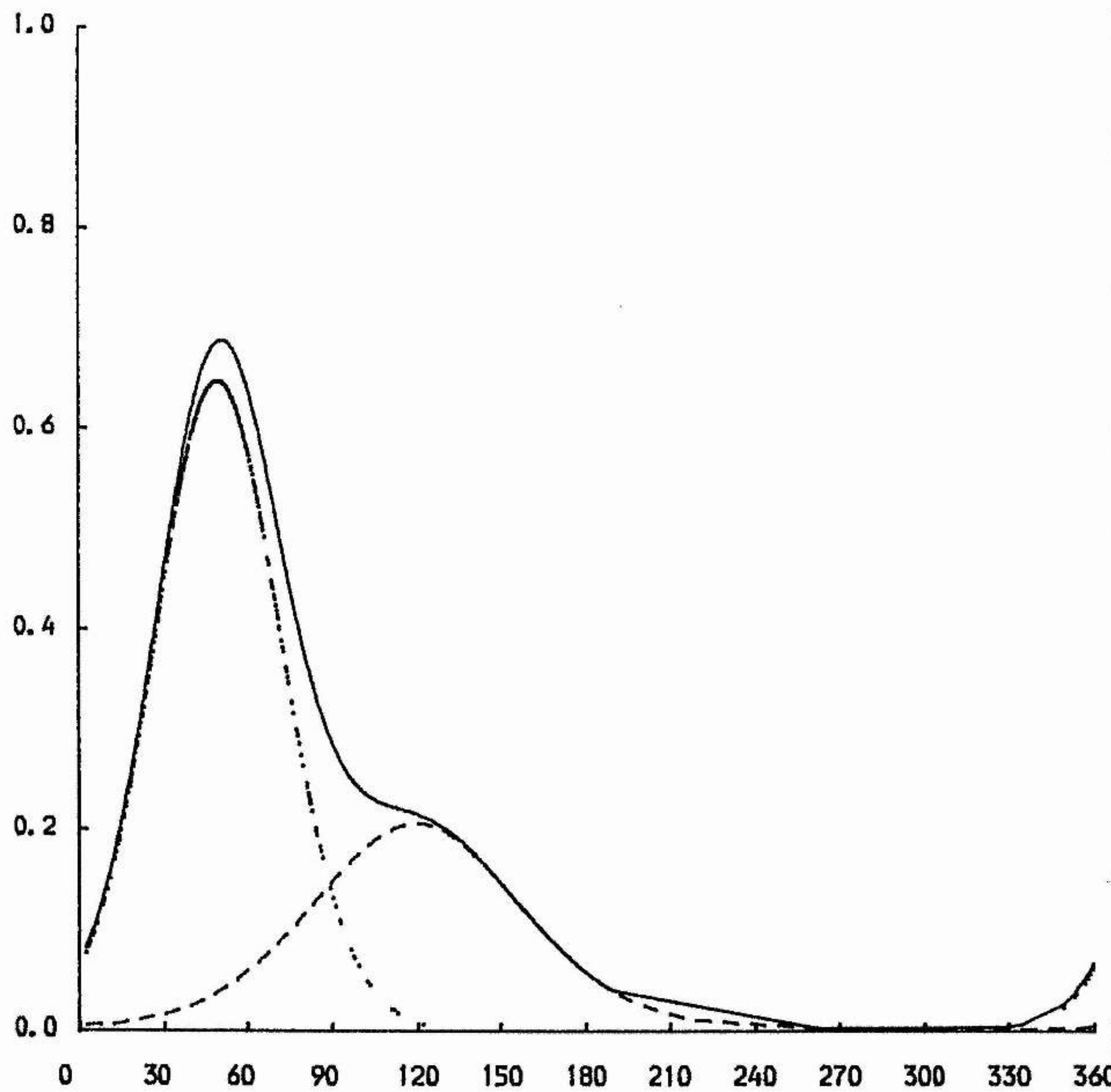


Figure (8)
The mixture density is displayed using the solid curve, and the underlying components p f_1 and (1 - p) f_2 are displayed using dotted and dash curves, respectively.
The true values are $p = 0.65$, $\kappa_1 = 6.5$, $\kappa_2 = 2.5$, $\mu_1 = 50.0$, $\mu_2 = 120.0$.

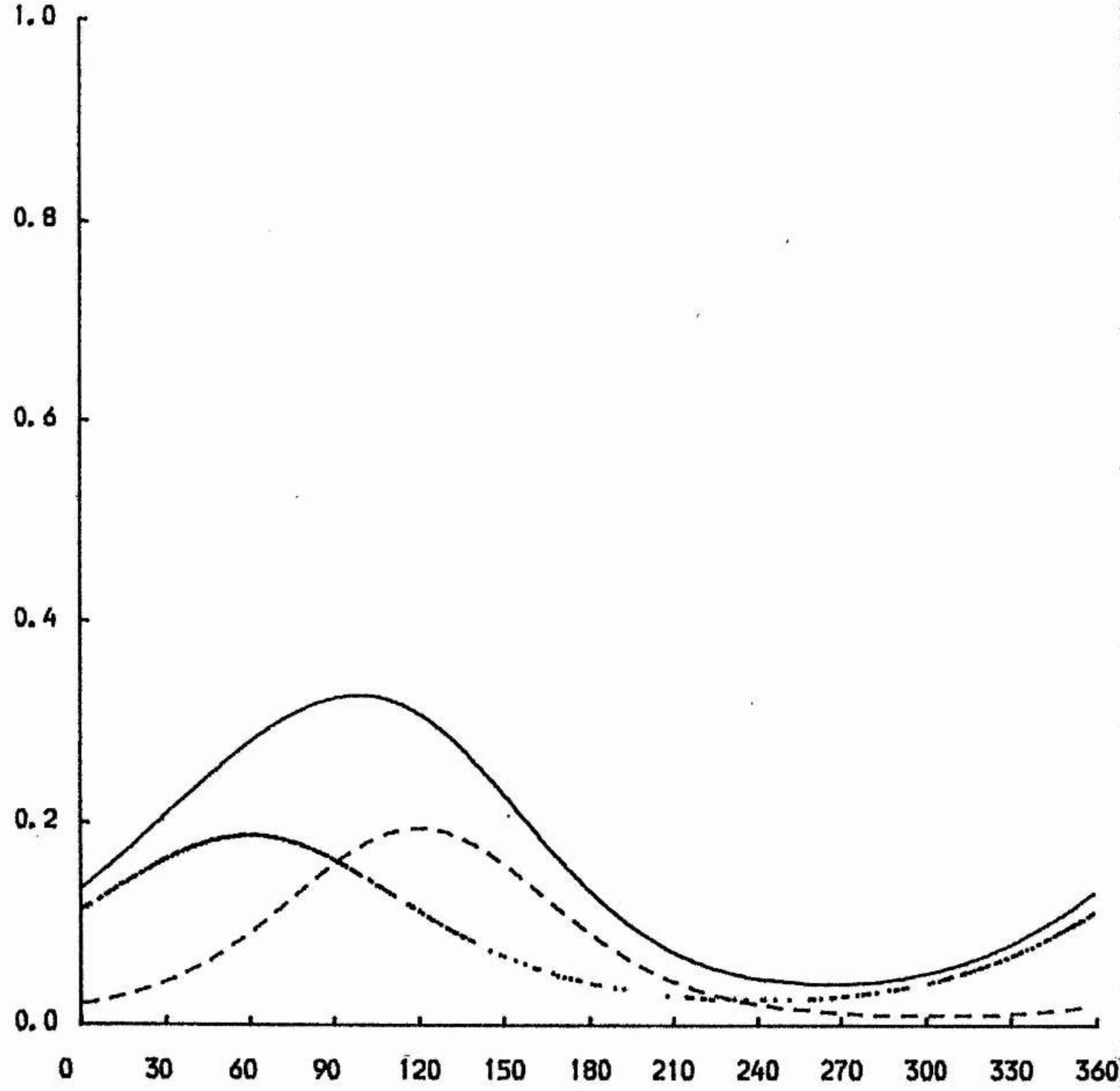


Figure (9)

The mixture density is displayed using the solid curve, and the underlying components $p\ f_1$ and $(1 - p)\ f_2$ are displayed using dotted and dash curves, respectively.

The true values are $p = 0.55$, $\kappa_1 = 1.0$, $\kappa_2 = 1.5$, $\mu_1 = 60.0$, $\mu_2 = 120.0$.

CHAPTER 3

METHODS OF ESTIMATION

3.1 Maximum Likelihood Estimation

In this section we introduce the maximum likelihood method for estimating the parameters of statistical distributions.

Let x_1 , x_2 , ..., x_n be independent observations from a density say , f(x ; α) , where α is the parameter vector we wish to estimate. The likelihood function is defined by

$$L(\alpha) = \prod_{j=1}^{n} f(x_j ; \alpha) . \qquad (3.1)$$

It measures the relative likelihood that different α will have given rise to the observed x's .

The maximum likelihood method finds the particular α say, α_0 which maximizes L , i.e the α_0 such that the observed x's are more likely to have come from f(x ; α_0) than f(x ; α) for any other value of α.

It is usually more convenient in practice to maximize

$$\lambda(\alpha) = \mathrm{Log}_e L(\alpha) = \sum_{j=1}^{n} \mathrm{Log}_e f(x_j ; \alpha) \qquad (3.2)$$

rather than maximizing L(α).

For many parameter estimation problems one can tackle this maximization in the traditional way of differentiating $\lambda(\alpha)$ with respect to the components of α and equating the derivatives to zero to give the normal equations

$$\frac{\partial\lambda}{\partial\alpha_i}=0 \tag{3.3}$$

These are then solved for the α_i and the second order derivatives are examined to verify that it is indeed a maximum which has been achieved and not some other stationary point.

For simple parametric models the maximum likelihood approach is very popular , partly because of the existence of attractive asymptotic theory , and partly because the estimates are often easy to compute.

For mixture distributions , however , the normal equations are not usually explicitly solvable and so iterative techniques have to be adopted.

The problem of obtaining the maximum likelihood estimates of the unknown parameters in the mixture of two linear normal distributions has been extensively studied in the literature by , amongst others , Hasselblad(1966) , Day(1969) , Wolfe(1970) , Hosmer(1973) and Fowlkes(1979). For further details see Titterington et al (1985).

Because of the nonlinear nature of the first partial derivatives of the log-likelihood function the maximum likelihood estimates cannot be obtained analytically. Consequently various iterative techniques have been proposed (see for example Hasselblad(1966) and Dempster, Laird and Rubin(1977) - the EM algorithm). The EM (Expectation - Maximization) algorithm given by Dempster , Laird and Rubin(1977) has become very popular recently and has been applied to many problems in statistics. Its main drawback is the slowness of convergence (see Everitt & Hand 1981) but this is compensated for by its simplicity and other properties. For further discussion and more details see Titterington et al (1985).

Unfortunately , whenever the two variance terms cannot be assumed equal , iterative numerical techniques may break down in practice (see Quandt and Ramsey(1978)).

For the mixture of two von Mises distributions the likelihood function is also unbounded and so iterative techniques are again necessary. Jones and James(1969) used a procedure formed from a combination of the gradient method and Newton - Raphson method. We use a modified Newton algorithm which is available as subroutine (E04JAF) of the NAG library.

Using this subroutine we sometimes have an overflow problem. If κ_1 or κ_2 become very large (greater than 80) then the Bessel function I_0 will cause an overflow . This tends to happen with small samples, when the mixing parameter p is very small and also when the concentration parameters κ_1 and κ_2 are small and almost equal . Putting bounds on κ_1 and κ_2 solves the problem but in practice one of them reaches the limit. Another way of getting round this problem is discussed later (see pages 49-51).

3.2 MOMENT ESTIMATION

Karl Pearson(1894) first used the method of moments to estimate the parameters in a mixture of two linear normal distributions. However , there are two major problems to overcome when the method of moments is applied to the mixture of two linear normal distributions. Firstly, we need to solve a high order polynomial equation and secondly , multiple solutions may exist. For an extensive bibliography and further details see Titterington et al (1985) and Everitt and Hand(1981).

We can adapt the method for the mixture of two von Mises distributions and make use of the sine and cosine moments . However , as noted by Mardia(1972) , there is the problem of selecting an appropriate set of trigonometric moments to estimate the five unknown parameters. When the first two sine and cosine moments are used there is no symmetrical way of constructing the fifth equation. For the special case when $\kappa_1 = \kappa_2$ and $\mu_2 = \mu_1 + \pi$ i.e. the three parameter case, Mardia(1972 pp. 128 - 129) described a way of obtaining estimates of the parameters using the method of moments. Spurr(1981) extended this method for data which are bimodal on the range $(0 , \pi)$.

We obtain four equations by taking the first two sine and cosine moments but choice of the fifth equation is arbitrary. For our method we based the fifth equation on the derivative of the log likelihood with respect to the parameter p.

The five moment equations are then

$$p\,\frac{I_t(\kappa_1)}{I0(\kappa_1)}\cos(t\,\mu_1) + (1-p)\frac{I_t(\kappa_2)}{I0(\kappa_2)}\cos(t\,\mu_2) = \frac{1}{n}\sum_{i=1}^{n}\cos(t\,\theta_i)$$

$$p\,\frac{I_t(\kappa_1)}{I0(\kappa_1)}\sin(t\,\mu_1) + (1-p)\frac{I_t(\kappa_2)}{I0(\kappa_2)}\sin(t\,\mu_2) = \frac{1}{n}\sum_{i=1}^{n}\sin(t\,\theta_i)\;, \tag{3.4}$$

for t =1,2,

and

$$\sum_{i=1}^{n}\frac{\{2\pi I0(\kappa_1)\}^{-1}\exp\{\kappa_1\cos(\theta_i-\mu_1)\} - \{2\pi I0(\kappa_2)\}^{-1}\exp\{\kappa_2\cos(\theta_i-\mu_2)\}}{p\{2\pi I0(\kappa_1)\}^{-1}\exp\{\kappa_1\cos(\theta_i-\mu_1)\} + (1-p)\{2\pi I0(\kappa_2)\}^{-1}\exp\{\kappa_2\cos(\theta_i-\mu_2)\}} = 0 \tag{3.5}$$

These equations were solved using a standard NAG minimisation subroutine. However , convergence to the final estimates using (3.4) and (3.5) can be very slow because equation (3.5) has to be calculated at each iteration. Also , the iterative procedure sometimes stops before convergence to the final estimates is achieved. The procedure can be restarted by using the last estimates given before stopping as the initial estimates in a new optimisation. This can be repeated until final convergence is attained.

Another method which is considerably quicker with regard to CPU time can be obtained by using the first three sine and cosine moments i.e. the six equations obtained from (3.4) with $t = 1, 2, 3$. This approach eliminates the lack of symmetry in the choice of the fifth equation and these equations can easily be solved using a standard NAG minimisation subroutine.

3.3 MINIMUM DISTANCE ESTIMATION

Minimum distance estimation was first introduced by Wolfowitz(1957) and the methods are usually based on distribution functions or transforms of the distribution function. Although less efficient than maximum likelihood when the assumed model is correct this method of estimation is more robust against heavy - tailed departures from the model (see Woodward et al (1984)). Also , Parr and Schucany(1980) showed that minimum distance techniques can provide robust estimators of the location parameter of a symmetric distribution.

The methods proposed so far in the literature fall into two main categories :

(i) measures based on the distance between the empirical and theoretical distribution functions ;
(ii) measures based on the distance between some transforms of the empirical and theoretical distribution functions.

As examples of (i) , Choi and Bulgren(1968) , MacDonald(1971) minimised the sum of squared distances between the empirical and theoretical distribution functions. Also , Woodward et al (1984) used a measure based on the Cramer-von Mises type distance between the empirical and theoretical distributions. They compared this method with maximum likelihood in a simulation study of a mixture of two normal components with all five parameters unknown. They were most interested in the estimation of the mixing parameter p. Their results showed that maximum likelihood was superior to their method when the assumed model was indeed normal but minimum distance gave better estimates under symmetric departures from normality.

Under (ii) , Quandt and Ramsey(1978) used a measure based on the squared distance between the empirical and theoretical moment generating functions. Heathcote(1977) introduced a measure based on the integrated squared error between the empirical and theoretical characteristic functions. For further details see Titterington et al (1985).

Methods of estimation based on the characteristic function are particularly useful for the mixture of two von Mises distributions since the characteristic function can be obtained in an analytic form and is always bounded. We also use a Cramer-von Mises type measure for comparative purposes but find that this is much slower and does not give estimates which are substantially better.

We now describe the two methods used beginning with the method based on the Cramer-von Mises measure.

(i) The Cramer-von Mises measure

This measure minimises

$$\sum_{i=1}^{n} \left[F(\theta_i) - \frac{(i-0.5)}{n}\right]^2 \tag{3.6}$$

where

$$F(\theta_i) = p \int_0^{\theta_i} \{2\pi I_0(\kappa_1)\}^{-1} \exp\{\kappa_1 \cos(x - \mu_1)\}\, dx$$

$$+ (1 - p) \int_0^{\theta_i} \{2\pi I_0(\kappa_2)\}^{-1} \exp\{\kappa_2 \cos(x - \mu_2)\}\, dx \tag{3.7}$$

and θ_i is the ith order statistic of the sample. Calculation of $F(\theta_i)$ is slow and considerably increases CPU time.

For the mixture of two linear normals Woodward et al (1984) obtained starting values by using an ad hoc quasi - clustering technique. In our case starting values are obtained using the procedure described in chapter 4 .

(ii) The characteristic function measure

The use of the characteristic function as the particular form of the transform has been discussed by , among others , Paulson , Holcomb , and Leitch(1975) , Bryant and Paulson(1979) , Feuerverger and McDunnough(1981) and Heathcote(1977). Since the characteristic function is always bounded and can be expressed in a closed form , we introduce a method based on the characteristic function for estimating the unknown parameters in a mixture of two von Mises distributions.

The characteristic function measure minimises

$$\sum_{t=1}^{\infty} \left| \Phi_n(t) - \Phi(t) \right|^2$$

where (3.8)

$$\Phi_n(t) = \frac{1}{n} \sum_{j=1}^{n} \exp(it\theta_j)$$

is the empirical characteristic function (ecf) of the sample θ_1 , θ_2 , ..., θ_n , and $\Phi(t)$ is the characteristic function of the model.

To find the characteristic function of (2.4) we must evaluate

$$\Phi(t) = E(e^{it\theta}) = E (\cos \theta t + i \sin \theta t)$$

Now

$$E(\cos\theta t) = \int_0^{2\pi} \cos(\theta t)\, f(\theta)\, d\theta$$

$$= p\,\{2\pi\, I_0(\kappa_1)\}^{-1} \int_0^{2\pi} \cos(\theta t)\, \exp\{\kappa_1 \cos(\theta - \mu_1)\}\, d\theta$$

$$+ (1 - p)\,\{2\pi\, I_0(\kappa_2)\}^{-1} \int_0^{2\pi} \cos(\theta t)\, \exp\{\kappa_2 \cos(\theta - \mu_2)\}\, d\theta$$

If we put $z = \theta - \mu_1$, $dz = d\theta$ we get

$$\int_0^{2\pi} \cos(\theta t)\, \exp\{\kappa_1 \cos(\theta-\mu_1)\}\, d\theta = \int_0^{2\pi} \cos\{(z+\mu_1)t\}\, \exp(\kappa_1 \cos z)\, dz$$

$$= \int_0^{2\pi} (\cos zt \cos \mu_1 t - \sin zt \sin \mu_1 t)\, \exp(\kappa_1 \cos z)\, dz$$

$$= \cos \mu_1 t \int_0^{2\pi} \cos(zt)\, \exp(\kappa_1 \cos z))\, dz$$

$$- \sin \mu_1 t \int_0^{2\pi} \sin zt\, \exp(\kappa_1 \cos z)\, dz$$

$$= 2\pi \cos(\mu_1 t)\, I_t(\kappa_1)\,.$$

Since $\int_0^{2\pi} \cos(zt) \exp(\kappa_1 \cos z)\, dz = 2\pi\, I_t(\kappa_1)$ and

$$\int_0^{2\pi} \sin zt \exp(\kappa_1 \cos z)\, dz = 0, \qquad \text{for } t = 1,2,.....$$

(see Mardia (1972, p .62)) .

The function $I_t(\kappa)$ is the modified Bessel function of the first kind and of order t .

Therefore

$$E(\cos \theta t) = p \cos(t\, \mu_1) \frac{I_t(\kappa_1)}{I_0(\kappa_1)} + (1 - p) \cos(t\, \mu_2) \frac{I_t(\kappa_2)}{I_0(\kappa_2)}$$

Similarly we have

$$E(\sin \theta t) = \int_0^{2\pi} \sin(\theta t)\, f(\theta)\, d\theta$$

$$= p\, \{2\pi\, I_0(\kappa_1)\}^{-1} \int_0^{2\pi} \sin(\theta t) \exp\{\kappa_1 \cos(\theta - \mu_1)\}\, d\theta$$

$$+ (1 - p)\, \{2\pi\, I_0(\kappa_2)\}^{-1} \int_0^{2\pi} \sin(\theta t) \exp\{\kappa_2 \cos(\theta - \mu_2)\}\, d\theta$$

with

$$\int_0^{2\pi} \sin(\theta t) \exp\{\kappa_1 \cos(\theta-\mu_1)\}\, d\theta = \int_0^{2\pi} \sin\{(z+\mu_1)t\} \exp(\kappa_1 \cos z)\, dz$$

$$= \int_0^{2\pi} (\sin zt \cos \mu_1 t + \cos zt \sin \mu_1 t) \exp(\kappa_1 \cos z)\, dz$$

$$= \cos \mu_1 t \int_0^{2\pi} \sin zt \exp(\kappa_1 \cos z)\, dz$$

$$+ \sin \mu_1 t \int_0^{2\pi} \cos zt \exp(\kappa_1 \cos z)\, dz$$

$$= 2\pi \sin \mu_1 t\, I_t(\kappa_1)$$

Therefore

$$E(\sin \theta t) = p \sin(t\,\mu_1) \frac{I_t(\kappa_1)}{I0(\kappa_1)} + (1 - p) \sin(t\,\mu_2) \frac{I_t(\kappa_2)}{I0(\kappa_2)}$$

Hence the characteristic function of (2.4) is given by

$$\Phi(t) = p\,\{\cos(t\,\mu_1) + i \sin(t\,\mu_1)\} \frac{I_t(\kappa_1)}{I0(\kappa_1)}$$

$$+ (1 - p)\,\{\cos(t\,\mu_2) + i \sin(t\,\mu_2)\} \frac{I_t(\kappa_2)}{I0(\kappa_2)} \qquad (3.9)$$

Paulson , Holcomb & Leitch(1975) developed a numerical method for estimating the parameters of the stable law by minimizing the integral

$$I = \int_{-\infty}^{\infty} \left|\Phi_n(t) - \Phi(t)\right|^2 e^{-t^2} dt \qquad (3.10)$$

They found that this estimation procedure worked well for some parameter values but failed to give reasonable results for other parameter values.

This method was generalised by Heathcote(1977). He introduced a measure based on the integrated squared error between the empirical and theoretical characteristic functions for estimating the parameters in a mixture of two ~~linear~~ normal distributions. Heathcote investigated the properties of the statistic θ_n which minimizes

$$I_n(\theta) = \int_{-\infty}^{\infty} \left|\Phi_n(t) - \Phi(t,\theta)\right|^2 dG(t) \qquad (3.11)$$

where $\Phi(t,\theta)$ is the theoretical characteristic function and G(t) is a nondecreasing weight function whose total variation can be taken as unity. The choice of weight function G(t) is very important with regard to the efficiency of the estimator and convergence of the method.

Heathcote discussed the choice of weight function for convergence and pointed out that the optimum weight function will generally depend on the distribution function. For example if θ is the variance of a normal distribution centred at the origin , then reasonable efficiency is achieved if the weight function assigns most weight to an interval about the origin. There may also be circumstances where it is preferable to minimize

$$\frac{1}{n}\sum \left|\Phi_n(t) - \Phi(t,\theta)\right|^2$$

at one , or only a few values of t rather than using a weight function distributed over the real line. He compared the integrated squared error estimate with the maximum likelihood estimate based on random samples of size n from a $N(0,\theta)$ distribution using the weight function $G'(t) = e^{-t^2}$. He found that maximum likelihood gave better results when the sample size was 25. When the sample sizes were 50 and 100 the results obtained from both methods were very similar. He also used the same weight function in the calculation of the integrated squared error estimate for the example of Cox & Hinkley(1974 , p . 291) for which the density is given by

$$f(x,\theta) = \frac{\varepsilon}{\sigma(2\pi)^{1/2}} \exp\left[-\frac{1}{2}\left\{\frac{x-\theta}{\sigma}\right\}^2\right] + \frac{(1-\varepsilon)}{(2\pi)^{1/2}} \exp\left[-\frac{1}{2}\left\{ x-\theta \right\}^2\right] .$$

For this example Heathcote found that the maximum likelihood estimator was inconsistent but the integrated squared error estimator was consistent.

For circular distributions , the characteristic function $\Phi(t)$ is only defined for integer values of t (see Mardia 1972) and consequently estimation using this measure is often more tractable than in the mixture of two linear normals.

Heathcote's method depends on the choice of t and weight function G(t) while this is less important for our method.

In order to minimise (3.8) we note that (3.9) involves Bessel functions of order t and these decay as t increases (see Kent 1977). Consequently we may not need to include too many terms in the summation for convergence to be achieved.

Equation (3.8) was minimised when the number of terms in the summation was taken as 2 , 5 , 8 and 10 respectively. The convergence when using 5 , 8 and 10 terms was satisfactory although CPU time increases as the number of terms increases. When only two terms are taken in the summation we find that the convergence is unstable , in other words when we use different initial values to start the minimisation procedure we get different final results. Hence we shall only consider using 5 , 8 and 10 terms in the summation.

For our method , convergence is achieved without using a weight function although we did use some weight functions to see what effect they had. Altogether we tried 5 different weight functions $\exp(-t^5)$,$\exp(-t)$,t^{-2}, t^{-1} chosen arbitrarily and $\exp(-t^2)$ used by Paulson , Holcomb, and Leitch(1975) and Heathcote(1977). We found that the only weight function which improved the convergence when using 5 , 8 and 10 terms in the summation was t^{-1}. When the weight functions $\exp(-t^5)$, $\exp(-t^2)$, $\exp(-t)$ and t^{-2} were used with 5 , 8 and 10 terms we found that the convergence was unstable.

An alternative expression for (3.8) can be obtained by substituting (3.9) into (3.8), expanding the square, and making use of the von Neumann addition formula which is given by

$$I_0\{(\kappa_1^2 + \kappa_2^2 + 2\kappa_1\kappa_2 \cos\theta)^{1/2}\} = I_0(\kappa_1)\, I_0(\kappa_2) + 2\sum_{t=1}^{\infty} I_t(\kappa_1) I_t(\kappa_2) \cos\theta t \qquad (3.12)$$

This leads to

$$\left|\Phi_n(t) - \Phi(t)\right|^2 = \left| \frac{1}{n}\sum_{j=1}^{n} \exp(it\theta_j) - p\,\{\cos(t\,\mu_1) + i\sin(t\,\mu_1)\}\frac{I_t(\kappa_1)}{I0(\kappa_1)} - (1-p)\,\{\cos(t\,\mu_2) + i\sin(t\,\mu_2)\}\frac{I_t(\kappa_2)}{I0(\kappa_2)}\right|^2$$

$$= \left| \left[\frac{1}{n}\sum_{j=1}^{n}\cos(t\theta_j) - \left\{p\cos(t\,\mu_1)\frac{I_t(\kappa_1)}{I0(\kappa_1)} + (1-p)\cos(t\,\mu_2)\frac{I_t(\kappa_2)}{I0(\kappa_2)}\right\}\right] + i\left[\frac{1}{n}\sum_{j=1}^{n}\sin(t\theta_j) - \left\{p\sin(t\,\mu_1)\frac{I_t(\kappa_1)}{I0(\kappa_1)} + (1-p)\sin(t\,\mu_2)\frac{I_t(\kappa_2)}{I0(\kappa_2)}\right\}\right]\right|^2$$

$$= \left[\frac{1}{n}\sum_{j=1}^{n}\cos(t\theta_j) - \left\{p\cos(t\,\mu_1)\frac{I_t(\kappa_1)}{I0(\kappa_1)} + (1-p)\cos(t\,\mu_2)\frac{I_t(\kappa_2)}{I0(\kappa_2)}\right\}\right]^2 + \left[\frac{1}{n}\sum_{j=1}^{n}\sin(t\theta_j) - \left\{p\sin(t\,\mu_1)\frac{I_t(\kappa_1)}{I0(\kappa_1)} + (1-p)\sin(t\,\mu_2)\frac{I_t(\kappa_2)}{I0(\kappa_2)}\right\}\right]^2$$

$$= [\ \bar{C} - (a_1 + a_2)\ ]^2 + [\ \bar{S} - (a_3 + a_4)\ ]^2$$

where

$$\bar{C} = \frac{1}{n}\sum_{j=1}^{n}\cos(t\theta_j)\ , \qquad a_1 = p\ \cos(t\ \mu_1)\frac{I_t(\kappa_1)}{I0(\kappa_1)}\ ,$$

$$a_2 = (1 - p)\ \cos(t\ \mu_2)\frac{I_t(\kappa_2)}{I0(\kappa_2)}\ , \qquad \bar{S} = \frac{1}{n}\sum_{j=1}^{n}\sin(t\theta_j)$$

$$a_3 = p\ \sin(t\ \mu_1)\frac{I_t(\kappa_1)}{I0(\kappa_1)} \quad \text{and} \quad a_4 = (1 - p)\ \sin(t\ \mu_2)\frac{I_t(\kappa_2)}{I0(\kappa_2)}.$$

Hence $\left|\Phi_n(t) - \Phi(t)\right|^2 = \bar{C}^2 - 2\bar{C}(a_1+a_2) + (a_1+a_2)^2 + \bar{S}^2$

$$- 2\ \bar{S}\ (a_3+a_4) + (a_3+a_4)^2$$

$$= \bar{C}^2 + \bar{S}^2 - 2\ \{\ \bar{C}\,(a_1+a_2) + \bar{S}\,(a_3+a_4)\ \} + (a_1+a_2)^2 + (a_3+a_4)^2$$

Now $(a_1+a_2)^2 = a_1^2 + a_2^2 + 2\ a_1a_2 = p^2\ \cos^2(t\ \mu_1)\left[\frac{I_t(\kappa_1)}{I0(\kappa_1)}\right]^2$

$$+ (1 - p)^2\ \cos^2(t\ \mu_2)\ \left[\frac{I_t(\kappa_2)}{I0(\kappa_2)}\right]^2$$

$$+ 2\ p\ (1 - p)\cos(t\ \mu_1)\cos(t\ \mu_2)\frac{I_t(\kappa_1)}{I0(\kappa_1)}\frac{I_t(\kappa_2)}{I0(\kappa_2)}$$

$$(a_3+a_4)^2 = a_3^2 + a_4^2 + 2a_3\, a_4 = p^2 \sin^2(t\,\mu_1)\left[\frac{I_t(\kappa_1)}{I0(\kappa_1)}\right]^2$$

$$+ (1-p)^2 \sin^2(t\,\mu_2)\left[\frac{I_t(\kappa_2)}{I0(\kappa_2)}\right]^2$$

$$+ 2\,p\,(1-p)\sin(t\,\mu_1)\sin(t\,\mu_2)\frac{I_t(\kappa_1)}{I0(\kappa_1)}\frac{I_t(\kappa_2)}{I0(\kappa_2)}$$

Therefore

$$(a_1+a_2)^2 + (a_3+a_4)^2 = p^2\left[\frac{I_t(\kappa_1)}{I0(\kappa_1)}\right]^2 + (1-p)^2\left[\frac{I_t(\kappa_2)}{I0(\kappa_2)}\right]^2$$

$$+ 2\,p\,(1-p)\cos t(\mu_1-\mu_2)\frac{I_t(\kappa_1)}{I0(\kappa_1)}\frac{I_t(\kappa_2)}{I0(\kappa_2)}$$

consider now

$\bar{C}\,(a_1+a_2) =$

$$\frac{1}{n}\sum_{j=1}^{n}\cos(t\theta_j)\left\{p\cos(t\,\mu_1)\frac{I_t(\kappa_1)}{I0(\kappa_1)} + (1-p)\cos(t\,\mu_2)\frac{I_t(\kappa_2)}{I0(\kappa_2)}\right\}$$

$\bar{S}(a_3+a_4) =$

$$\frac{1}{n}\sum_{j=1}^{n}\sin(t\theta_j)\left\{p\sin(t\,\mu_1)\frac{I_t(\kappa_1)}{I0(\kappa_1)} + (1-p)\sin(t\,\mu_2)\frac{I_t(\kappa_2)}{I0(\kappa_2)}\right\}$$

Then $\bar{C}\,(a_1+a_2) + \bar{S}\,(a_3+a_4) = p\dfrac{I_t(\kappa_1)}{I0(\kappa_1)}\,\dfrac{1}{n}\sum_{j=1}^{n}\cos t(\theta_j-\mu_1)$

$$+ (1-p)\frac{I_t(\kappa_2)}{I0(\kappa_2)}\,\frac{1}{n}\sum_{j=1}^{n}\cos t(\theta_j-\mu_2)$$

$$\bar{C}^2 + \bar{S}^2 = \left\{\frac{1}{n}\sum_{j=1}^{n}\cos(t\theta_j)\right\}^2 + \left\{\frac{1}{n}\sum_{j=1}^{n}\sin(t\theta_j)\right\}^2 = \bar{R}_t^2 \text{ say .}$$

Collecting these together we have

$$\sum_{t=1}^{\infty}\left|\bar{C} - (a_1 + a_2) + i\,(\bar{S} - (a_3 + a_4))\right|^2$$

$$= \sum_{t=1}^{\infty}\bar{R}_t^2 - \frac{2p}{n}\sum_{t=1}^{\infty}\frac{I_t(\kappa_1)}{I_0(\kappa_1)}\sum_{j=1}^{n}\cos t(\theta_j - \mu_1) -$$

$$\frac{2(1-p)}{n}\sum_{t=1}^{\infty}\frac{I_t(\kappa_2)}{I_0(\kappa_2)}\sum_{j=1}^{n}\cos t(\theta_j - \mu_2) + p^2\sum_{t=1}^{\infty}\left[\frac{I_t(\kappa_1)}{I_0(\kappa_1)}\right]^2 +$$

$$(1-p)^2\sum_{t=1}^{\infty}\left[\frac{I_t(\kappa_2)}{I_0(\kappa_2)}\right]^2 + 2p(1-p)\sum_{t=1}^{\infty}\cos t(\mu_1 - \mu_2)\frac{I_t(\kappa_1)}{I_0(\kappa_1)}\frac{I_t(\kappa_2)}{I_0(\kappa_2)}$$

Using (3.12) we then have

$$\sum_{t=1}^{\infty}I_t(\kappa_1)I_t(\kappa_2)\cos t(\mu_1 - \mu_2) =$$

$$\frac{1}{2}[I_0\{(\kappa_1^2 + \kappa_2^2 + 2\kappa_1\kappa_2\cos(\mu_1 - \mu_2))^{1/2}\}] - \frac{1}{2}I_0(\kappa_1)\,I_0(\kappa_2)$$

Therefore

$$2p\,(1-p)\sum_{t=1}^{\infty}\cos\, t(\mu_1-\mu_2)\,\frac{I_t(\kappa_1)}{I_0(\kappa_1)}\,\frac{I_t(\kappa_2)}{I_0(\kappa_2)} =$$

$$p\,(1-p)\,\left[\frac{I_0\{(\kappa_1{}^2+\kappa_2{}^2+2\kappa_1\kappa_2\cos(\mu_1-\mu_2))^{1/2}\}}{I_0(\kappa_1)\,I_0(\kappa_2)}-1\right]$$

and

$$\sum_{t=1}^{\infty}\left[\frac{I_t(\kappa_2)}{I_0(\kappa_2)}\right]^2=\frac{1}{[I_0(\kappa_2)]^2}\left[\frac{1}{2}I_0\{(\kappa_2{}^2+\kappa_2{}^2+2\kappa_2{}^2)^{1/2}\}-\frac{1}{2}\{I_0(\kappa_2)\}^2\right]$$

(von Neumann formula with $\kappa_1=\kappa_2$, $\mu_1=\mu_2$) .

Also $e^{z\cos\theta}=I_0(z)+2\sum_{m=1}^{\infty}I_m(z)\cos(m\theta)$

(see Abramowitz & Stegun p . 376)

Therefore

$$\sum_{t=1}^{\infty}I_t(z)\cos\, t(\theta_j-\mu_1)=\frac{1}{2}\,\exp(z\cos(\theta_j-\mu_1))-\frac{1}{2}I_0(z)$$

Whence

$$\sum_{t=1}^{\infty} \left|\Phi_n(t) - \Phi(t)\right|^2 = \sum_{t=1}^{\infty} \bar{R}_t^2 - \frac{p}{nI_0(\kappa_1)}\sum_{j=1}^{n}\exp\{\kappa_1 \cos(\theta_j - \mu_1)\}$$

$$- \frac{(1-p)}{nI_0(\kappa_2)}\sum_{j=1}^{n}\exp\{\kappa_2 \cos(\theta_j - \mu_2)\} + \frac{p^2}{2}\frac{I_0(2\kappa_1)}{[I_0(\kappa_1)]^2}$$

$$+ \frac{(1-p)^2}{2}\frac{I_0(2\kappa_2)}{[I_0(\kappa_2)]^2} + p\,(1-p)\left[\frac{I_0\{(\kappa_1{}^2 + \kappa_2{}^2 + 2\kappa_1\kappa_2 \cos(\mu_1 - \mu_2))^{1/2}\}}{I_0(\kappa_1)\, I_0(\kappa_2)}\right]$$

$$+ \frac{1}{2} \qquad (3.13)$$

The first and last terms on the right hand side of (3.13) do not depend on the parameters to be estimated and so can be omitted from the optimisation routine. All the Bessel functions have been reduced to order zero and no longer depend on t. Hence we no longer have an infinite sum in t to contend with.

We use a standard subroutine (S18AEF) of the NAG library to calculate the Bessel function I_0. There is an overflow problem with the Bessel function when the argument is greater than 80 and this is caused by either κ_1 or κ_2 becoming very large in the iterative optimisation procedure. Setting upper bounds for the κ's prevents an overflow but in practice one of the κ's reaches the limit. This can happen for any initial values . One way round this difficulty is to use the following large κ approximation to I_0

$$I_0(\kappa_i) \cong \frac{\exp(\kappa_i)}{(2\pi\kappa_i)^{1/2}} \left\{ 1 + \frac{1}{8\kappa_i} \right\} \qquad i = 1, 2 \qquad (3.14)$$

This gives

$$\frac{I_0(2\kappa_i)}{[I_0(\kappa_i)]^2} = \frac{(4\pi\kappa_i)^{-1/2}\ \{\ 1 + \frac{1}{16\kappa_i}\ \}\ 2\pi\kappa_i}{\{\ 1\ + \frac{1}{8\kappa_i}\ \}^2}\ , i = 1, 2 \qquad (3.15)$$

To simplify $\dfrac{I_0\{(\kappa_1{}^2 + \kappa_2{}^2 + 2\kappa_1\kappa_2\cos(\mu_1 - \mu_2))^{1/2}\}}{I_0(\kappa_1)\, I_0(\kappa_2)}$

using (3.14) we have the following .

If $S = \{\kappa_1{}^2 + \kappa_2{}^2 + 2\kappa_1\kappa_2\cos(\mu_1 - \mu_2)\}^{1/2}$ is big then either κ_1 and κ_2 are big and so

$$\frac{I_0(S)}{I_0(\kappa_1)\, I_0(\kappa_2)} = \frac{\exp(S-\kappa_1-\kappa_2)(2\pi\kappa_1)^{1/2}(2\pi\kappa_2)^{1/2}\{1 + \frac{1}{8S}\ \}}{(2\pi S)^{1/2}\{1 + \frac{1}{8\kappa_1}\}\{1 + \frac{1}{8\kappa_2}\}}$$

or κ_1 is big and κ_2 is small and so

$$\frac{I_0(S)}{I_0(\kappa_1)\, I_0(\kappa_2)} = \frac{\exp(S-\kappa_1)(2\pi\kappa_1)^{1/2}\{1 + \frac{1}{8S}\}}{(2\pi S)^{1/2}\{1 + \frac{1}{8\kappa_1}\}}\frac{1}{I_0(\kappa_2)}$$

[The term $I_0(\kappa_2)$ can then be calculated by using the NAG Library subroutine .]

or κ_2 is big and κ_1 is small and so

$$\frac{I_0(S)}{I_0(\kappa_1)\, I_0(\kappa_2)} = \frac{\exp(S-\kappa_2)(2\pi\kappa_2)^{1/2}\{1 + \frac{1}{8S}\}}{(2\pi S)^{1/2}\{1 + \frac{1}{8\kappa_2}\}}\frac{1}{I_0(\kappa_1)}$$

In which case $I_0(\kappa_1)$ can be calculated by using the NAG Library subroutine.

Finally we can write

$$\frac{1}{I_0(\kappa_i)}\sum_{j=1}^{n}\exp\{\kappa_i\cos(\theta_j - \mu_i)\} =$$

$$\sum_{j=1}^{n}\exp[-\kappa_i\{1-\cos(\theta_j - \mu_i)\}]\,\frac{(2\pi\kappa_i)^{1/2}}{[1+\frac{1}{8\kappa_i}]}$$

for large κ_i ($i = 1, 2$) .

Using this approximation for I_0 improves the convergence but it does not work for all samples and so we still have samples where one of the κ's reaches the upper limit. This seriously affects the bias and MSE calculations. For further discussion and detailed results of simulations see Chapter 4.

CHAPTER 4

DISCUSSION OF RESULTS

In this chapter we demonstrate our results with two kinds of data :

(1) Simulated data.

(2) Real data introduced in Section(4.2).

The minimisation subroutines require initial values i.e. starting values for the five parameters μ_1 , μ_2 , κ_1 , κ_2 and p.

Mardia(1972) gave an explicit method for estimating the parameters p , κ and μ for the three parameter model (2.5). Our starting value procedure uses this method to obtain estimates of p , κ_1 and μ_1. We then set $\kappa_2 = \kappa_1$ and $\mu_2 = \mu_1 + \pi$ to obtain starting values for κ_2 and μ_2.

The procedure is as follows :-

(1) estimate μ_1 from the equation

$$\tan(2\mu_1) = \sum_{i=1}^{n} \sin(2\theta_i) \Big/ \sum_{i=1}^{n} \cos(2\theta_i) \tag{4.1}$$

where θ_1 , θ_2 , ..., θ_n are the observed angles ;

(2) estimate κ_1 from the equation

$$\frac{I_2(\kappa_1)}{I_0(\kappa_1)} = \frac{1}{n}\left[\left(\sum_{i=1}^{n} \cos(2\theta_i)\right)^2 + \left(\sum_{i=1}^{n} \sin(2\theta_i)\right)^2\right]^{1/2} = \bar{R}_2 \tag{4.2}$$

(3) p can then be estimated from the equation

$$(2p - 1)\, A(\kappa_1) = \cos(\mu_1)\frac{1}{n}\sum_{i=1}^{n}\cos(\theta_i) + \sin(\mu_1)\frac{1}{n}\sum_{i=1}^{n}\sin(\theta_i) \qquad (4.3)$$

where $A(\kappa_1) = \dfrac{I_1(\kappa_1)}{I_0(\kappa_1)}$. Values of $A(\kappa_1)$ are tabulated in Appendix 2.2 of Mardia(1972) and also in Table 3 of Spurr(1981). For large κ_1 we have $A(\kappa_1) \cong 1 - \dfrac{1}{2\kappa_1}$, and for small κ_1 , $A(\kappa_1) \cong \dfrac{\kappa_1}{2} - \dfrac{\kappa_1^{3}}{16}$ (see Appendix 1 of Mardia(1972)).

Equation (4.2) is difficult to invert for κ_1 although selected values are given in Table 3 of Spurr(1981). However, since

$\dfrac{I_2(\kappa_1)}{I_0(\kappa_1)} = 1 - 2\left\{ \dfrac{A(\kappa_1)}{\kappa_1} \right\}$ then we can make use of the above approximations to $A(\kappa_1)$.

For large κ_1 equation (4.2) becomes

$$1 - 2\left\{ \frac{1}{\kappa_1}\left(1 - \frac{1}{2\kappa_1} \right) \right\} \cong \bar{R}_2$$

i.e.

$$\kappa_1 \cong \frac{2}{(1 - \bar{R}_2)} . \qquad (4.4)$$

For small κ_1 equation (4.2) becomes

$$1 - 2\left\{ \frac{1}{\kappa_1}\left(\frac{\kappa_1}{2} - \frac{\kappa_1^{3}}{16} \right) \right\} \cong \bar{R}_2$$

i.e.

$$\kappa_1 \cong (8 \ \bar{R}_2)^{1/2} \ . \tag{4.5}$$

We suggest using approximation (4.4) for values of $\bar{R}_2 > 0.5$

and approximation (4.5) for $\bar{R}_2 < 0.5$.

These approximations are not very accurate for $\bar{R}_2$ near 0.5 but since we are only interested in obtaining starting values for an iterative procedure this does not matter too much.

4.1 The Simulated data:

We have two ways of comparing the results for the different methods of estimation discussed earlier.

(i) The first way is to compare the results for single samples of different sizes i.e. n = 50,100 and 200. The reason for doing this is , firstly, to see how close the estimates of the unknown parameters get to the true values and, secondly, to see how much CPU time is taken. However, because of sampling variations we would need to look at many samples to get an overall view of the relative performances of the various methods.

(ii) The second way is to generate many samples of the same size and compare the bias and MSE for the different methods. This way will give us a better overall idea of the relative merits of the different methods.

We represent maximum likelihood estimation by MLE, moment estimation using five equations by MME1, moment estimation using six equations by MME2, minimum distance estimation based on the Cramer-von Mises measure by MDE1 and minimum distance estimation based on the simplified characteristic function measure (3.13) by MDE2. Minimum distance estimation when using 5, 8 and 10 terms in the summation (3.8) without a weight function will be denoted by sum5, sum8 and sum10 and with a weight function by sum5W, sum8W and sum10W.

Firstly then, for single samples, we compare the results obtained using the various methods with regard to CPU time and agreement to the true values.

Tables 1 - 6 display the results for the six simulated data sets which we shall use throughout this section. These are :

Example 1: $p = 0.35, \kappa_1 = 3.5, \kappa_2 = 6.0, \mu_1 = 100, \mu_2 = 320$

Example 2: $p = 0.70, \kappa_1 = 7.5, \kappa_2 = 1.5, \mu_1 = 120, \mu_2 = 340$

Example 3: $p = 0.10, \kappa_1 = 5.0, \kappa_2 = 2.5, \mu_1 = 75, \mu_2 = 210$

Example 4: $p = 0.20, \kappa_1 = 2.0, \kappa_2 = 3.0, \mu_1 = 100, \mu_2 = 230$

Example 5: $p = 0.65, \kappa_1 = 6.5, \kappa_2 = 2.5, \mu_1 = 50, \mu_2 = 120$

Example 6: $p = 0.55, \kappa_1 = 1.0, \kappa_2 = 1.5, \mu_1 = 60, \mu_2 = 120.$

These simulations were generated using the method described in Section(2.6).

Each method of estimation discussed in Chapter 3 was implemented for the above mixtures.

We compared the final results obtained firstly when the true values were used as initial values and secondly when the initial values were obtained from the procedure described at the beginning of this chapter.

In practice, of course, we will never know the true values but the purpose here is to show that the final results do not depend on the initial values used.

If either $\quad | \hat{p}(\text{Tru}) - \hat{p}(\text{Ini}) | < 0.01$

$$\tag{4.6}$$

or $\quad | \hat{p}(\text{Tru}) - 1 + \hat{p}(\text{Ini}) | < 0.01$

where

$\hat{p}(\text{Tru})$ is the estimate of p when the true values are used as initial values , $\hat{p}(\text{Ini})$ is the estimate of p when the initial values are obtained from (4.1) - (4.3), then we consider the two results to be in good agreement.

We need both conditions in (4.6) since sometimes p and (1-p) are interchanged in the final results with consequent changes to the other parameters.

Generally, the estimates of μ_1 and μ_2 are close to the true values (see tables 1,2,3,5) although there is some variability between the different methods of estimation. In table 4 some of the methods give values of μ_1 which are very different from the true values although they get closer as n increases. We have problems with example 6 (see table 6) since most of the parameter estimates are not close to the true values. This is to be expected since the mean directions are close together, the concentration parameters are almost equal and small and as shown in figure (9) the distribution is unimodal.

The estimates of κ_1 and κ_2 are very variable, often one of them will be reasonably close to the true value but the other will generally be much bigger. Usually they get closer as the sample size increases but not always. There are big differences between the different methods here with no one method consistently estimating κ_1 and κ_2 better than the others.

The estimates of p also vary considerably from method to method. Again, as n increases the estimates generally get closer to the true values. As we might expect the estimates of p are better when the concentrations are large and the modes are separated.

From these tables, it does not seem that one method produces consistently better results than any of the others. However, in most cases the CPU time taken when using MME2 and sum5W are similar and these methods are the fastest over all. Also, MME1 and MDE1 take a large amount of CPU time and do not give substantially better estimates than the other methods. Consequently we decided not to include them in the bias and MSE comparisons.

Increasing the sample size usually improves the accuracy. Also, the use of the weight function does not consistently improve the results.

Secondly, we consider some comparative measures based on sets of samples namely the bias and MSE. We simulated 50 and 200 samples of size 50 , 100, 200 for each of the six examples given earlier.

In order to compare the different methods of estimation discussed earlier the bias and MSE for each parameter were calculated for each set of samples. Samples which did not satisfy condition (4.6) were noted but excluded from the bias and MSE calculations. Samples where one of the κ's reached the upper limit were also excluded.

For each of the six examples we have 6 sets of results i.e. 50 and 200 samples of sizes 50, 100 and 200.

Each of the six sets of results is described by a single table which compares sum5, sum8 and sum10 with sum5W, sum8W and sum10W to find out which seems to be the best. Also included are the results for MME2, MDE2 and MLE for comparison with this best method.

Each table gives the bias and MSE for each of the parameters and indicates the number of samples satisfying condition (4.6), C1, and the number of samples which converge but do not satisfy condition (4.6), C2. The number of samples where one of the κ's reaches the upper limit is denoted by C3.

It is clear from tables 7-12 of example 1 and tables 13-18 of example 2 that the use of the weight function reduces the bias and MSE for all the parameters. In addition the number satisfying condition (4.6) is increased. There is little difference between sum8W and sum10W as far as bias and MSE are concerned but sum5W generally does better in this respect. All the samples converged when sum5W , sum8W and sum10W were used but condition (4.6) was not always satisfied. However, there is only one case where this condition is not satisfied by sum5W. Also, the average CPU time taken when using sum5W is less than that when using either sum8W or sum10W. Consequently we choose sum5W as our best method. In other words we need take no more than 5 terms in the summation to ensure convergence and the smallest bias and MSE.

The tables also show that in most cases there is little to choose between MLE, MME2 and sum5W although MDE2 does not perform as well. MLE is slightly better than MME2 and sum5W in most cases but is much better than MDE2. MLE always converged but did not always satisfy condition (4.6) . Also, the average CPU time taken when using MLE is greater than that when using either sum5W or MME2 but is similar to that when using MDE2. Thus if reduced CPU time is our main criterion we may prefer to use MME2 or sum5W.

Tables 19-24 give the results for example 3. These tables show that using the weight function does not always reduce the bias or the MSE for both sum8 and sum10 but most of the bias and MSE are reduced for sum5. In addition the number of samples satisfying condition (4.6) is increased when the weight function is used. In most cases sum5W performs better than sum8W and sum10W with regard to bias and MSE. The number of samples which satisfy condition (4.6) when using sum5W is always greater than that when using sum10W and, apart from one case (table 20), is always greater than or equal to that when using sum8W. Also, the average CPU time taken when using sum5W is less than that when using either sum8W or sum10W. Consequently we again choose sum5W as our best method. The tables also show that MLE performs better than MME2 , sum5W and MDE2 in almost all cases. MLE always converged but did not always satisfy condition (4.6). Also , MLE gave a greater number of samples satisfying condition (4.6) than MME2 , MDE2 for all cases but in some cases sum5W did better. However , the average CPU time taken when using MLE is greater than that taken when using either sum5W or MME2 but is similar to that taken when using MDE2.

Tables 25-30 give the results for example 4. From tables 26, 29 and 30 we see that the use of the weight function reduces the bias and MSE for sum5 , sum8 and sum10. However , tables 25, 27 and 28 show that using the weight function does not always reduce the bias or the MSE for some of the parameters. In fact the results are very variable , sometimes the bias is reduced and MSE increased or vice versa. In addition the number of samples satisfying condition (4.6) is increased when the weight function is used. In most of these tables it is clear that sum10W performs slightly better than sum5W and sum8W as far as bias and MSE are concerned. However , the number of samples which satisfy condition (4.6) when using sum5W is greater than that when using either sum8W or sum10W. Also, the CPU time taken when using sum5W is less than that when using either sum8W or sum10W. For these reasons we choose sum5W as our best method.

The tables also show that in most cases there is little to choose between MME2 and MLE which in turn are better than sum5W and much better than MDE2. In many cases MME2 performs slightly better than MLE. Also, the average CPU time taken when using MLE is greater than that taken when using either sum5W or MME2 but is similar to that taken when using MDE2.

Tables 31-36 show the results for example 5. From these tables we see that use of the weight function reduces the bias and MSE for most of the cases considered. In addition the number of samples satisfying condition (4.6) is increased. In most cases it seems that sum5W performs slightly better than sum8W and much better than sum10W as far as bias and MSE are concerned. However , sum8W usually does better with regard to the number of samples satisfying condition (4.6). Since the CPU time taken when using sum5W is less than that when using either sum8W or sum10W we again choose sum5W to be our best method.

The tables also show that , in almost all cases , MME2 performs better than MLE , sum5W and much better than MDE2. Generally there is little to choose between MME2 , MLE and sum5W as far as the number of samples satisfying condition (4.6) is concerned. Clearly MDE2 does not perform as well as the other methods. The average CPU time taken when using MME2 is similar to that taken when using sum5W but is less than that when using any of the other methods.

Tables 37-42 show the results for example 6. In most cases the bias and MSE are reduced when the weight function is used. In addition the number of samples which satisfy condition (4.6) is increased. For this example , the average CPU time taken when using either sum10 or sum10W is excessive and the results do not seem to be substantially better than sum 5 or sum8 (see table 37). Consequently , we decided not to include either sum10 or sum10W in the other tables for this example. From the tables it is clear that , in most cases, sum5W performs better than sum8W as far as bias and MSE are concerned. Also the number of samples satisfying condition (4.6) when using sum5W is greater in most cases than when using sum8W . Since the average CPU time taken when using sum5W is less than that when using sum8W we again choose sum5W as our best method.

The tables also show that , in most cases, MME2 performs better than MLE , sum5W and MDE2 as far as bias and MSE are concerned. However, the number of samples which satisfy condition (4.6) when using MLE is much greater than that when using MME2 , sum5W and MDE2. This will affect the bias and MSE calculations and may explain why MLE does not do well here. Also, the average CPU time taken when using MLE is greater than that when using either MME2 or sum5W but is similar to MDE2.

When the sample size n is increased the bias and MSE are reduced in almost every case for all the examples except example 1. For example 1 (tables 7-12) , as sample size is increased the MSE is reduced in almost all cases but the bias seems to be very variable for some parameters. For examples 1- 5 the number of samples satisfying condition (4.6) is increased whilst the number where one of the κ's reaches the limit is decreased. For example 6 there are one or two cases where the number of samples satisfying condition (4.6) is decreased as n is increased. Increasing the number of samples S usually leads to a slight increase in the bias and MSE and also an increase in the number C2 of samples where one of the κ's reaches the upper limit. However, there is considerable variability both between and within the examples used. As noted earlier example 6 is very different to the other examples and this is shown by the results. There are also big differences between the magnitudes of the bias and MSE for the different parameters.

In conclusion we find that MLE, MME2 and sum5W seem to perform best overall, for the examples considered, although MLE takes more CPU time for convergence than the others.

Table 1

n = 50

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.3961	8.1523	5.5031	99.9763	317.5181	0.93
sum8	0.37577	11.0286	5.0175	100.2130	317.3858	1.43
sum10	0.3683	12.3763	4.9147	100.1888	317.3727	1.57
sum5w	0.3892	8.0791	4.9893	99.5977	316.7052	1.31
sum8w	0.3832	9.3283	4.7926	99.7806	316.7490	6.78
sum10w	0.3822	9.5790	4.7688	99.7945	316.7590	7.25
MME1	0.3836	7.3253	4.2423	99.2404	315.5838	15.03
MME2	0.3776	10.1019	4.3857	98.6314	316.6393	0.72
MDE1	0.3816	7.6267	4.7480	99.2662	315.9961	19.03
MDE2	0.3613	13.9269	4.8148	100.0927	317.3607	6.50
MLE	0.3825	7.7616	4.0803	98.4053	315.0984	7.10

n = 100

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.3657	5.7891	6.1375	97.1296	318.0015	1.47
sum8	0.4393	3.8136	10.8319	104.0203	324.3592	1.51
sum10	0.4392	3.8158	10.8260	104.0199	324.3954	1.60
sum5w	0.3620	2.8917	6.2417	98.2011	316.3398	1.06
sum8w	0.3608	5.9576	5.8770	97.0198	317.7576	1.74
sum10w	0.3606	5.9743	5.8680	97.0245	317.7603	2.03
MME1	0.3564	6.3796	5.5090	96.9402	317.4967	28.04
MME2	0.3550	6.9591	5.5724	96.7238	317.8051	1.16
MDE1	0.3473	5.8736	5.3823	99.0807	318.0807	33.48
MDE2	0.3617	5.8163	5.8056	97.2668	317.9811	10.44
MLE	0.3558	6.6763	5.3424	96.3952	317.2158	10.97

n = 200

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.3941	4.0837	5.2387	98.9912	318.1771	1.22
sum8	0.3934	4.0223	5.1496	99.1300	318.2298	1.27
sum10	0.3933	4.0250	5.1448	99.1305	318.2311	1.29
sum5w	0.3021	3.3718	5.9261	103.5092	320.5620	0.94
sum8W	0.3621	3.6590	6.3785	98.4583	317.3888	1.66
sum10W	0.3927	4.1836	5.2314	98.9571	317.9947	11.59
MME1	0.3874	4.6963	5.1065	99.1952	318.0838	151.13
MME2	0.3905	4.4612	5.222	98.9616	317.8199	0.91
MDE1	0.3814	4.1707	4.8011	101.5129	318.8763	70.67
MDE2	0.3933	3.9974	5.1225	99.1362	318.2244	19.35
MLE	0.3873	4.7381	5.1033	99.1467	318.0392	20.78

The true values are $p = 0.35$, $\kappa_1 = 3.5$, $\kappa_2 = 6.0$, $\mu_1 = 100.0$, $\mu_2 = 320.0$.

Table 2

n = 50

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.6595	8.2778	1.2088	121.0635	334.6224	1.13
sum8	0.8205	4.3110	3.1221	121.0132	337.5828	2.1
sum10	0.6593	8.2132	1.2011	122.1564	335.9401	6.41
sum5W	0.6465	8.7978	1.1646	120.5014	335.3056	1.29
sum8W	0.6747	10.3005	2.0326	117.4586	333.3315	1.90
sum10W	0.6737	10.4008	2.0173	117.4613	333.4026	2.13
MME1	0.6564	8.0622	1.2346	118.8101	330.4019	18.45
MME2	0.6116	12.2061	0.9019	119.9827	338.1534	1.36
MDE1	0.6522	7.0558	1.1445	119.5870	331.5777	17.19
MDE2	0.6602	8.1414	1.2086	122.1241	335.7564	5.34
MLE	0.6485	9.5901	1.2244	119.4102	332.2022	6.13

n = 100

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.6862	8.5233	1.3160	119.7171	337.4105	1.38
sum8	0.6730	9.0524	1.1965	120.9663	340.0546	4.40
sum10	0.7873	5.0285	3.2871	118.5344	337.3740	1.84
sum5W	0.6642	7.5134	1.1148	115.4213	316.9730	1.24
sum8W	0.6691	7.2741	1.1541	115.3844	316.4450	1.60
sum10W	0.6693	7.2669	1.1554	115.3761	316.4113	2.13
MME1	0.6762	8.6248	1.2341	118.7111	335.2695	23.84
MME2	0.6516	11.0671	1.0496	118.9374	339.4577	1.24
MDE1	0.6781	7.3573	1.1858	118.0328	330.4956	35.25
MDE2	0.6719	9.0609	1.1895	121.0829	340.2594	10.86
MLE	0.6716	9.6885	1.1799	118.2027	333.6571	11.50

n = 200

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.7309	6.7481	1.3079	118.6955	328.8135	1.01
sum8	0.7337	6.6100	1.3243	119.4052	329.5861	1.93
sum10	0.7342	6.5919	1.3281	119.4430	329.5655	6.97
sum5W	0.7209	7.0838	1.2973	118.4835	329.6974	1.08
sum8W	0.7231	6.9491	1.3236	118.7764	330.1480	1.61
sum10W	0.7234	6.9359	1.3261	118.7874	330.1453	2.11
MME1	0.7278	6.9423	1.4228	117.6722	326.6865	43.80
MME2	0.6938	8.4420	1.0204	119.0877	333.2860	0.54
MDE1	0.7321	5.8988	1.3035	118.1735	327.0270	70.05
MDE2	0.7333	6.5815	1.3259	119.4561	329.5466	15.24
MLE	0.7273	7.5165	1.4871	117.9097	326.8847	15.70

The true values are $p = 0.70$, $\kappa_1 = 7.5$, $\kappa_2 = 1.5$, $\mu_1 = 120.0$, $\mu_2 = 340.0$.

Table 3

n = 50

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.1889	13.3567	3.4865	78.9900	212.4464	1.13
sum8	0.1884	12.1553	3.4252	75.9966	212.4754	2.04
sum10	0.1736	18.9579	3.3244	74.9187	212.3651	8.76
sum5W	0.1934	11.8390	3.5122	80.1076	211.5277	1.34
sum8W	0.1937	10.6401	3.4916	78.9544	211.4829	1.94
sum10W	0.1912	11.8128	3.4620	78.5222	211.4016	1.84
MME1	0.1936	11.0708	3.6229	82.9600	211.3808	44.15
MME2	0.2020	8.1745	3.6247	82.4262	211.5900	1.21
MDE1	0.1834	7.4678	3.2227	84.6041	212.2295	16.87
MDE2	0.1846	13.6868	3.3977	75.7101	212.4426	5.65
MLE	0.1869	12.6427	3.4235	80.2719	209.9762	5.90

n = 100

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.1444	19.1635	2.7789	83.8613	209.5114	1.54
sum8	0.1193	7.5616	2.4530	65.5093	209.7601	2.33
sum10	0.0796	5.6120	2.5839	79.9938	211.3593	8.69
sum5W	0.1554	13.7567	2.8643	83.7044	208.6753	1.54
sum8W	0.1624	9.1911	2.9181	83.8819	208.9152	1.86
sum10W	0.8366	1.9152	74.4239	199.0342	221.5274	2.77
MME1	0.1603	10.2290	2.9512	83.3524	208.1279	38.42
MME2	0.1811	5.8962	3.2004	87.7698	209.3019	1.03
MDE1	0.1580	6.5634	2.8751	87.3590	209.8632	33.13
MDE2	0.1571	9.8702	2.8333	82.9223	209.7561	10.53
MLE	0.1606	10.0409	2.9164	84.0177	208.1060	10.20

n = 200

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.1059	14.3651	2.3216	80.5322	211.2128	1.52
sum8	0.1284	5.4267	2.4251	77.8669	211.6681	1.54
sum10	0.1281	5.4758	2.4236	77.8622	211.6609	9.89
sum5W	0.1262	7.0910	2.4497	78.2321	210.4463	1.76
sum8W	0.1851	2.2243	2.6671	88.7761	206.0374	1.82
sum10W	0.13337	5.1907	2.4984	78.5574	210.7131	5.40
MME1	0.1303	7.0489	2.5504	77.6449	209.8685	63.13
MME2	0.1592	2.9180	2.6920	82.6295	211.5618	0.94
MDE1	0.1027	11.4870	2.0830	87.6035	211.0857	83.27
MDE2	0.1930	2.1003	2.7247	89.7913	206.0383	19.27
MLE	0.1325	5.9109	2.5562	77.0578	209.6421	19.90

The true values are $p = 0.10$, $\kappa_1 = 5.0$, $\kappa_2 = 2.5$, $\mu_1 = 75.0$, $\mu_2 = 210.0$.

Table 4

n = 50

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.4416	0.8467	5.5649	149.0888	238.8562	1.29
sum8	0.4343	0.8863	5.3472	148.9369	239.6214	2.46
sum10	0.4293	0.8956	5.2759	147.7479	239.5285	6.71
sum5W	0.4408	0.8368	5.5837	148.0037	238.3489	0.83
sum8W	0.4292	0.8714	5.3371	145.6888	238.5153	1.51
sum10W	0.4278	0.8740	5.3135	145.3613	238.4904	4.86
MME1	0.1685	3.8955	2.4595	91.6634	230.1023	23.71
MME2	0.4500	0.7920	5.9459	148.4639	237.5393	0.94
MDE1	0.2743	1.6225	3.6344	113.5436	235.9504	17.58
MDE2	0.4292	0.8954	5.2471	147.2976	239.5028	3.96
MLE	0.4674	0.7395	6.0807	154.8490	237.2572	4.74

n = 100

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.4092	1.0124	4.6000	142.5593	238.9059	1.19
sum8	0.1197	2.6387	2.9235	69.0743	225.4546	1.42
sum10	0.1197	2.6387	2.9235	69.0782	225.4555	2.50
sum5W	0.3919	1.0727	4.4671	138.1002	238.0251	1.05
sum8W	0.2373	1.9937	2.9492	102.1496	228.8925	1.55
sum10W	0.2373	1.9935	2.9494	102.1515	228.8928	2.52
MME1	0.1704	4.6885	2.4280	96.2828	229.7738	40.33
MME2	0.3452	1.2675	4.0662	127.2768	236.2145	1.15
MDE1	0.3429	1.3323	4.0848	128.0386	238.1489	36.68
MDE2	0.1197	2.6385	2.9236	69.0796	225.4557	9.09
MLE	0.3756	1.0281	4.1538	134.7024	236.2507	9.74

n = 200

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.2516	1.3383	3.0296	113.7962	233.0938	1.24
sum8	0.3255	0.9288	3.4501	128.1272	227.8986	1.28
sum10	0.3254	0.9292	3.4494	128.1013	227.8976	7.25
sum5W	0.2488	1.4304	3.0718	112.5157	232.3092	1.18
sum8W	0.2509	1.4150	3.0798	113.1702	232.4371	1.45
sum10W	0.2509	1.4151	3.0798	113.1690	232.4370	2.96
MME1	0.2035	2.0608	2.8560	101.1100	230.2612	149.10
MME2	0.2425	1.4349	3.0349	110.0881	231.9043	1.20
MDE1	0.2089	1.8043	2.8072	111.1975	232.4200	119.69
MDE2	0.2606	1.2739	3.0718	110.4499	233.5532	13.35
MLE	0.2071	1.9699	2.8679	102.0305	230.3565	14.86

The true values are $p = 0.20$, $\kappa_1 = 2.0$, $\kappa_2 = 3.0$, $\mu_1 = 100.0$, $\mu_2 = 230.0$.

Table 5

n = 50

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.8190	5.3092	13.5146	55.4532	138.3366	1.07
sum8	0.8239	5.1272	14.0602	55.7076	140.9773	1.60
sum10	0.8641	3.3234	17.2367	62.7948	146.3518	2.69
sum5W	0.8097	5.3814	10.5746	55.1449	139.0757	1.19
sum8W	0.8199	5.1537	13.1216	55.5545	140.6282	2.02
sum10W	0.8303	4.9697	18.0688	55.9395	142.0885	2.12
MME1	0.7568	5.5895	3.7085	53.1158	131.0657	4.49
MME2	0.7668	5.7273	4.3913	53.3850	133.3979	0.98
MDE1	0.8117	5.0676	4.5958	55.6835	145.4499	18.68
MDE2	0.8504	4.7412	34.4766	56.0989	142.7170	4.41
MLE	0.7694	5.7204	4.4838	53.2608	134.5885	4.60

n = 100

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.6958	4.9779	3.3061	51.3882	131.6500	0.87
sum8	0.8351	5.6008	13.9319	53.8474	136.1134	1.51
sum10	0.8583	5.2391	28.4280	54.1272	139.6344	2.21
sum5W	0.8551	5.1829	32.9907	53.5192	137.5937	1.32
sum8W	0.8437	5.3961	15.9794	53.3004	136.6948	1.58
sum10W	0.8514	5.2565	21.9744	53.5630	138.0469	2.02
MME1	0.7948	5.4685	4.2783	50.9259	126.5221	46.91
MME2	0.8380	5.2493	8.5707	52.7639	136.1564	1.17
MDE1	0.8178	5.5879	3.1713	54.2275	146.4680	35.44
MDE2	0.8634	5.1851	38.2881	54.1092	137.7117	7.21
MLE	0.8152	5.6469	6.6304	51.3122	132.9853	7.92

n = 200

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.7912	5.3255	3.3897	49.8767	129.9371	0.90
sum8	0.8415	4.5058	12.0793	54.6178	139.5505	1.49
sum10	0.8575	4.3180	19.0261	54.9092	141.0504	2.35
sum5W	0.6540	6.0686	3.2760	51.4614	125.7004	0.88
sum8W	0.8408	4.4896	11.9339	54.3027	140.7007	1.54
sum10W	0.8463	4.4184	14.6727	54.4926	141.1289	2.00
MME1	0.7573	5.0250	3.0768	51.0577	126.4606	120.05
MME2	0.7843	5.2362	4.5617	51.7520	132.7143	0.93
MDE1	0.8269	4.5291	3.8728	55.0677	148.4469	110.55
MDE2	0.8661	4.2245	26.8218	54.9390	139.6121	15.22
MLE	0.7493	5.5624	3.0025	50.5790	127.1414	15.25

The true values are $p = 0.65$, $\kappa_1 = 6.5$, $\kappa_2 = 2.5$, $\mu_1 = 50.0$, $\mu_2 = 120.0$

Table 6

n = 50

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.2683	4.3274	0.5788	81.3164	108.5799	1.44
sum8	0.7640	2.3755	8.1612	50.9078	143.0420	1.79
sum10	0.7577	2.4105	7.3873	50.8269	143.9148	12.45
sum5W	0.4989	3.6892	1.3348	43.6260	130.7860	1.25
sum8W	0.5366	3.3560	1.4555	44.4601	135.0687	1.44
sum10W	0.5276	3.3914	1.2376	44.4231	136.1280	13.30
MME1	0.6998	0.4566	2.2175	57.4298	109.5379	2.96
MME2	0.7020	0.5415	5.2208	48.2810	119.8711	1.57
MDE1	0.8590	0.8900	18.8272	72.0521	156.3336	91.06
MDE2	0.5510	0.5622	3.0479	85.7117	103.6515	6.04
MLE	0.2015	5.8444	0.6667	80.6735	102.2470	6.56

n = 100

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.2008	1.0622	1.3753	6.7263	96.8803	1.77
sum8	0.8220	1.4545	6.6654	53.9283	147.2928	2.72
sum10	0.8206	1.4595	6.5309	53.8831	147.4883	17.87
sum5W	0.2147	1.0762	1.4198	6.1070	97.5197	2.06
sum8W	0.2151	1.0742	1.4207	86.0748	97.5264	3.71
sum10W	0.2063	1.0775	1.3590	86.0512	97.6711	19.50
MME1	0.3811	0.7961	1.7949	16.3271	102.8884	34.99
MME2	0.7617	0.8851	2.7224	65.9674	118.7161	1.13
MDE1	0.0705	24.5916	1.1784	16.6666	98.3466	177.89
MDE2	0.2017	1.0579	1.3776	6.7053	96.8862	11.20
MLE	0.3802	0.8238	1.8339	14.1982	103.4548	10.88

n = 200

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.8620	0.9790	6.9311	65.9149	146.3685	2.43
sum8	0.9171	1.0007	44.0614	77.4373	156.5281	3.25
sum10	0.0446	55.8022	1.2243	13.7629	102.5809	27.86
sum5W	0.8575	0.9858	6.3219	66.0811	147.1823	5.30
sum8W	0.8183	1.0178	4.3072	61.9946	144.9253	10.82
sum10W	0.3553	52.7620	1.4450	35.9845	108.1168	35.22
MME1	0.5930	1.2117	1.0590	63.8489	134.4842	55.69
MME2	0.9342	0.9749	5.3041	81.2528	171.8157	1.57
MDE1	0.9234	0.9753	10.8668	79.1478	161.6749	189.10
MDE2	0.9178	1.0047	36.7561	77.7551	158.3542	27.94
MLE	0.9291	0.9259	47.9311	80.1987	158.0217	28.50

The true values are $p = 0.55$, $\kappa_1 = 1.0$, $\kappa_2 = 1.5$, $\mu_1 = 60.0$, $\mu_2 = 120.0$.

EXAMPLE 1

Table 7 : n=50 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	- 0.0078	0.0039	1.8083	19.1285	0.6100	7.1405	0.0314	124.2082	0.6670	19.5100	50	0	0
sum8	- 0.0037	0.0045	2.7804	49.9425	0.9379	12.1854	1.0456	107.9803	0.6034	22.3784	47	2	1
sum10	- 0.0029	0.0046	2.6972	46.9601	1.0212	13.2692	0.9911	110.7997	0.5921	22.5278	47	3	0
sum5W	- 0.0028	0.0033	0.9988	8.0932	0.5511	5.8500	0.6381	84.6208	0.5427	16.5458	50	0	0
sum8W	- 0.0009	0.0032	1.0966	11.0565	0.7623	8.0673	0.4955	86.3317	0.5997	17.4320	50	0	0
sum10W	- 0.0004	0.0032	1.0450	9.4793	0.7968	8.3836	0.4589	87.0313	0.6002	17.4828	50	0	0
MME2	0.0011	0.0036	0.9708	8.3128	0.4782	3.6034	0.0757	102.2035	0.1588	15.9226	50	0	0
MDE2	0.0051	0.0058	2.7844	114.6854	1.2497	15.0235	- 0.6158	294.6270	0.8378	22.5453	45	1	4
MLE	- 0.0029	0.0033	1.1081	11.0898	0.5974	5.7004	0.4026	84.2863	0.3150	15.2261	50	0	0

EXAMPLE 1

Table 8 : n=50 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	- 0.0029	0.0052	2.0953	17.3101	0.6445	5.8457	- 0.0521	113.9083	- 0.2694	24.7739	200	0	0
sum8	- 0.0005	0.0066	2.9237	39.5090	1.1438	16.4941	- 0.1086	135.5307	- 0.3921	28.5882	197	2	1
sum10	0.0005	0.0068	2.9250	39.8421	1.3172	23.7975	- 0.0773	141.3720	- 0.4013	28.8899	196	4	0
sum5W	- 0.0003	0.0049	1.5712	10.2758	0.5909	4.9198	- 0.0536	91.5872	- 0.2056	21.9953	200	0	0
sum8W	0.0004	0.0050	1.6131	11.1476	0.7536	7.6018	- 0.1205	90.7839	- 0.2085	22.8431	200	0	0
sum10W	0.0006	0.0050	1.6085	10.9326	0.7824	8.1586	- 0.1301	90.9829	- 0.2068	22.8757	200	0	0
MME2	0.0017	0.0051	1.3943	8.6150	0.5648	3.9926	- 0.0747	92.6647	- 0.3603	20.5435	200	0	0
MDE2	0.0004	0.0060	3.0905	64.5340	1.0853	17.7490	- 0.0546	157.4253	- 0.3896	28.4646	184	7	9
MLE	- 0.0007	0.0049	1.6425	11.8844	0.5869	4.1683	0.0130	86.3227	- 0.0619	20.6085	200	0	0

EXAMPLE 1

Table 9 : n=100 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	- 0.0018	0.0016	0.6872	3.3573	0.0860	2.4072	0.3296	35.5565	- 0.1892	9.4390	50	0	0
sum8	- 0.0004	0.0017	0.8246	4.9928	0.2460	3.6753	0.2190	38.5748	- 0.2642	10.4964	50	0	0
sum10	- 0.0001	0.0017	0.8094	4.7281	0.2709	3.9478	0.1900	39.1824	- 0.2687	10.5982	50	0	0
sum5W	- 0.0008	0.0015	0.5976	2.8735	0.0768	2.1961	0.1726	34.9036	- 0.1381	8.1881	50	0	0
sum8W	- 0.0002	0.0015	0.5826	2.7012	0.1286	2.5358	0.0942	35.6541	- 0.1663	8.5027	50	0	0
sum10W	- 0.0002	0.0015	0.5825	2.6893	0.1317	2.5650	0.0948	35.7130	- 0.1665	8.5120	50	0	0
MME2	0.0005	0.0015	0.5369	2.9288	0.0667	2.1090	0.1450	36.0443	- 0.2793	8.0715	50	0	0
MDE2	0.0005	0.0019	0.9206	6.9464	0.3252	4.5662	0.1385	41.3384	- 0.3527	10.6275	49	1	0
MLE	- 0.0021	0.0016	0.5742	1.9080	0.0455	1.8060	0.3143	35.3130	- 0.0289	7.7388	50	0	0

EXAMPLE 1

Table 10 : n=100 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	-0.0008	0.0024	0.7809	3.4899	0.1882	1.9946	-0.3698	46.9863	-0.3779	12.0811	200	0	0
sum8	-0.0006	0.0026	0.9712	5.6677	0.2924	2.8495	-0.3971	49.6891	-0.4413	13.0371	200	0	0
sum10	-0.0002	0.0027	0.9543	5.5753	0.3232	3.1438	-0.4056	49.5900	-0.4405	13.0853	200	0	0
sum5W	0.0004	0.0023	0.6687	2.7470	0.1854	1.7747	-0.4909	42.6897	-0.2976	10.7980	200	0	0
sum8W	0.0004	0.0024	0.7130	3.1640	0.2059	1.9733	-0.5066	43.3454	-0.3170	11.0093	200	0	0
sum10W	0.0005	0.0024	0.7133	3.1836	0.2099	2.0024	-0.5078	43.3563	-0.3165	11.0131	200	0	0
MME2	0.0013	0.0024	0.5613	2.2241	0.2194	1.8691	-0.3795	43.8191	-0.3587	10.7966	200	0	0
MDE2	0.0011	0.0028	0.9830	6.6629	0.3585	3.6635	-0.4247	50.1893	-0.5130	13.0622	197	3	0
MLE	0.0002	0.0022	0.6083	1.8223	0.2468	1.6461	-0.4391	40.1971	-0.1589	10.1009	200	0	0

EXAMPLE 1

Table 11 : n=200 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	- 0.0053	0.0010	0.4911	1.1513	0.0279	1.0757	1.2531	35.5091	0.4833	4.4916	50	0	0
sum8	- 0.0031	0.0011	0.4472	1.1904	0.1741	1.6502	1.2247	35.7345	0.4227	5.0584	50	0	0
sum10	- 0.0025	0.0011	0.4853	1.1490	0.1307	1.5704	1.3481	35.9369	0.4696	5.1007	49	1	0
sum5W	- 0.0046	0.0009	0.4256	0.9701	0.0074	0.8934	1.1499	30.6876	0.4920	3.9775	50	0	0
sum8W	- 0.0039	0.0010	0.4047	0.9812	0.0721	1.1346	1.1052	30.7205	0.4626	4.1119	50	0	0
sum10W	- 0.0039	0.0010	0.4047	0.9793	0.0717	1.3350	1.1059	30.7015	0.4630	4.1125	50	0	0
MME2	- 0.0035	0.0009	0.4052	1.1225	0.0012	0.7572	1.0071	31.5111	0.4345	3.8282	50	0	0
MDE2	- 0.0035	0.0011	0.4274	1.1647	0.1709	1.7051	1.2641	36.2812	0.3713	5.0164	49	1	0
MLE	- 0.0046	0.0009	0.4062	0.8812	- 0.0435	0.4904	1.1493	25.8701	0.5547	3.5614	50	0	0

EXAMPLE 1

Table 12 : n=200 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	- 0.0049	0.0013	0.3914	1.0410	0.0120	0.9820	0.2802	27.7305	0.0447	6.3201	200	0	0
sum8	- 0.0047	0.0014	0.4418	1.4171	0.0602	1.3077	0.2929	28.0308	0.0189	6.8103	200	0	0
sum10	- 0.0045	0.0014	0.4507	1.4220	0.0535	1.3101	0.3181	28.0393	0.0290	6.8358	199	1	0
sum5W	- 0.0041	0.0012	0.3303	0.7813	0.0155	0.8709	0.2065	24.3769	0.0692	5.6946	200	0	0
sum8W	- 0.0041	0.0013	0.3455	0.8601	0.0281	0.9875	0.2040	24.4569	0.0611	5.8105	200	0	0
sum10W	- 0.0041	0.0013	0.3454	0.8645	0.0289	0.9913	0.2039	24.4517	0.0612	5.8126	200	0	0
MME2	- 0.0031	0.0012	0.2825	0.7666	0.0602	0.9163	0.2014	25.0384	0.0172	5.5758	200	0	0
MDE2	- 0.0045	0.0014	0.4313	1.5493	0.0710	1.3686	0.2820	28.5671	0.0021	6.9073	195	5	0
MLE	- 0.0035	0.0012	0.2952	0.6142	0.0765	0.7228	0.2049	21.5947	0.1157	5.2221	200	0	0

EXAMPLE 2

Table 13 : n=50 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	- 0.0142	0.0065	1.0409	9.8999	1.7816	32.7142	1.0249	20.7498	5.1934	697.4218	48	2	0
sum8	- 0.0165	0.0115	1.8545	21.5025	2.9554	114.0028	1.3275	27.6898	4.529	646.0336	46	4	0
sum10	- 0.0100	0.0084	1.4865	18.8057	2.8998	126.1481	0.9178	24.2584	3.6996	839.2753	40	10	0
sum5W	- 0.0120	0.0060	0.8406	9.6164	0.7768	4.0162	0.7904	18.7209	2.5673	450.8716	49	1	0
sum8W	- 0.0151	0.0077	1.1442	11.3418	0.7190	2.7901	0.9989	21.1194	3.4009	494.9672	49	1	0
sum10W	- 0.0107	0.0073	1.0898	11.3059	1.0524	8.9349	0.8161	19.9058	3.3800	550.4518	47	3	0
MME2	- 0.0072	0.0083	1.5827	44.1482	0.7786	2.5819	0.6674	16.5616	0.9225	505.0143	49	0	1
MDE2	- 0.0454	0.0198	4.1803	113.8699	2.5629	116.3357	0.6161	32.0025	5.1526	978.3379	48	0	2
MLE	- 0.0136	0.0065	1.2915	14.6061	0.5146	1.5505	0.4217	15.9336	4.2362	556.7668	48	2	0

EXAMPLE 2

Table 14 : n=50 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	- 0.0027	0.0082	1.4971	12.6367	1.5283	14.7721	- 0.1705	21.1159	2.3301	650.0053	196	3	1
sum8	- 0.0033	0.0105	2.0707	24.7264	2.4399	53.0073	- 0.0793	24.4206	3.0677	741.5863	176	22	2
sum10	0.0058	0.0097	1.9677	24.8050	2.7792	61.3959	- 0.3409	22.59997	- 0.8395	601.1541	154	45	1
sum5W	- 0.0020	0.0074	1.3268	10.9972	0.9011	4.3766	- 0.2614	19.2285	1.7037	579.9955	199	1	0
sum8W	- 0.0043	0.0082	1.7178	17.3188	0.9657	4.6013	- 0.1636	20.5768	1.6367	520.2365	189	11	0
sum10W	0.0045	0.0068	1.4777	14.9360	1.2439	7.2417	- 0.2046	19.4022	- 0.1535	429.4424	175	25	0
MME2	- 0.0018	0.0088	2.3166	46.5191	0.9914	5.1519	- 0.2628	17.6046	1.6322	659.3941	198	0	2
MDE2	- 0.0224	0.0138	3.5667	86.9215	1.9496	59.2024	- 0.3019	25.8732	5.8916	1035.9535	181	4	15
MLE	- 0.0021	0.0077	1.6026	19.6363	0.6378	1.9451	- 0.3382	17.8762	0.8490	539.3016	194	6	0

EXAMPLE 2

Table 15 : n=100 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0006	0.0028	- 0.0357	3.3160	0.6427	2.5281	- 0.1144	9.5811	- 1.3711	199.3173	50	0	0
sum8	- 0.0037	0.0033	0.0790	4.5074	0.6954	3.8453	- 0.0579	11.1088	- 1.13711	244.1902	49	1	0
sum10	- 0.0006	0.0032	0.0317	4.5708	0.6926	2.7813	- 0.2183	10.7911	- 2.8240	169.0547	47	3	0
sum5W	- 0.0019	0.0025	- 0.0425	2.8379	0.3402	0.7116	- 0.1511	9.2284	- 2.4619	173.9713	50	0	0
sum8W	- 0.0022	0.0024	- 0.0043	3.3726	0.3257	0.6789	- 0.1296	9.6274	- 2.4506	176.3889	50	0	0
sum10W	- 0.0022	0.0024	- 0.0035	3.3909	0.3255	0.6789	- 0.1185	9.6326	- 2.4517	176.4215	50	0	0
MME2	- 0.0063	0.0042	0.5901	10.3283	0.3713	0.8228	- 0.1983	10.1178	- 1.3986	238.5822	50	0	0
MDE2	- 0.0045	0.0033	0.1462	4.4855	0.5966	2.6626	- 0.0212	10.9718	- 0.8854	237.9587	50	0	0
MLE	- 0.0036	0.0025	0.1409	3.7718	0.2649	0.5553	- 0.3542	9.9068	- 3.1340	181.1649	50	0	0

EXAMPLE 2

Table 16 : n=100 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	- 0.0025	0.0037	0.5054	3.7644	0.4653	1.5473	0.3789	10.6485	0.7451	272.2262	200	0	0
sum8	- 0.0061	0.0048	0.7103	6.0207	0.5302	2.4225	- 0.3729	11.9585	0.8890	315.7436	192	8	0
sum10	0.0036	0.0036	0.5536	5.3669	0.6512	2.1511	- 0.4252	11.9747	- 1.7693	168.0917	171	29	0
sum5W	- 0.0030	0.0035	0.4623	3.3544	0.3447	0.9276	- 0.4033	10.0166	0.3789	254.9528	200	0	0
sum8W	- 0.0030	0.0030	0.5459	4.2506	0.3654	1.0227	- 0.4283	10.5988	- 0.2984	218.4319	197	3	0
sum10W	- 0.0006	0.0032	0.4847	4.1123	0.3981	1.0204	- 0.3843	10.7432	- 0.9741	192.8205	191	9	0
MME2	- 0.0064	0.0047	0.9239	9.4929	0.3967	1.1937	- 0.3538	10.0816	1.6958	379.1699	200	0	0
MDE2	- 0.0102	0.0056	1.0460	14.6789	0.4403	1.9305	- 0.3210	12.3945	2.3463	368.6130	197	2	1
MLE	- 0.0024	0.0034	0.5264	3.7163	0.2848	0.6618	- 0.4736	9.4386	- 0.3035	212.3674	199	1	0

EXAMPLE 2

Table 17 : n=200 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	- 0.0031	0.0017	- 0.0759	1.3765	0.1546	0.3838	0.2446	4.6618	- 0.5752	80.9488	50	0	0
sum8	- 0.0032	0.0018	- 0.0136	1.9477	0.1292	0.2752	0.2527	5.2969	- 0.8987	80.0909	49	1	0
sum10	- 0.0022	0.0018	- 0.0683	2.1193	0.1485	0.2583	0.1725	4.8946	- 1.0133	77.1787	48	2	0
sum5W	- 0.0038	0.0016	- 0.0718	1.3045	0.1087	0.2508	0.1729	4.5312	- 0.6953	76.2211	50	0	0
sum8W	- 0.0027	0.0016	- 0.0923	1.5339	0.1077	0.2137	0.1901	4.7887	- 0.8776	75.7154	50	0	0
sum10W	- 0.0027	0.0016	- 0.0904	1.5485	0.1068	0.2116	0.1905	4.7987	- 0.8807	75.5974	50	0	0
MME2	- 0.0063	0.0021	0.1641	2.7522	0.1132	0.3653	0.1795	4.3544	- 0.3824	92.6160	50	0	0
MDE2	- 0.0037	0.0017	- 0.0475	2.0264	0.1143	0.2502	0.2990	5.3502	- 0.8430	78.4523	50	0	0
MLE	- 0.0030	0.0015	- 0.0171	1.5495	0.1210	0.2448	0.0490	4.0979	- 1.0506	78.0656	50	0	0

EXAMPLE 2

Table 18 : n=200 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	- 0.0027	0.0020	0.1941	1.4893	0.1677	0.4446	- 0.1791	5.2327	0.5843	121.7838	200	0	0
sum8	- 0.0037	0.0022	0.2678	2.1481	0.1566	0.4403	- 0.1704	5.9410	0.7634	128.7824	199	1	0
sum10	- 0.0020	0.0021	0.1993	2.2158	0.1946	0.4322	- 0.2044	5.8541	0.0881	112.5586	189	11	0
sum5W	0.0032	0.0020	0.1845	1.4427	0.1308	0.3529	- 0.1743	4.9769	0.5492	117.4608	200	0	0
sum8W	- 0.0029	0.0020	0.1873	1.6733	0.1315	0.3472	- 0.1687	5.2309	0.5328	118.6378	200	0	0
sum10W	- 0.0030	0.0020	0.1941	1.7228	0.1303	0.3460	- 0.1673	5.2406	0.5493	119.1196	200	0	0
MME2	- 0.0049	0.0025	0.4079	3.9814	0.1625	0.4858	- 0.1272	4.9462	1.2103	160.5378	200	0	0
MDE2	- 0.0059	0.0023	0.3155	2.6275	0.1255	0.3938	- 0.1275	5.9286	0.9954	134.3612	199	1	0
MLE	- 0.0033	0.0019	0.2062	1.5104	0.0988	0.2500	- 0.1657	4.4409	0.4408	107.5921	200	0	0

EXAMPLE 3

Table 19 : n=50 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.1685	0.0765	1.2785	47.1105	5.2774	206.1527	26.8380	3376.6164	2.3877	102.7851	28	6	16
sum8	0.0746	0.0319	7.6706	218.1446	0.9440	3.7988	19.1417	1891.4365	4.4698	71.9937	24	15	11
sum10	0.1032	0.0496	2.9773	47.8398	4.9623	242.4547	10.6944	1452.1845	2.1061	107.9319	22	16	12
sum5W	0.1596	0.0674	0.8762	38.3491	3.0536	33.5458	29.8138	3598.7201	3.9610	118.7136	37	3	10
sum8W	0.1371	0.0603	6.0949	306.7623	2.4894	24.0037	20.5266	2613.0966	3.1931	110.6159	34	6	10
sum10W	0.1232	0.0491	1.5825	40.6671	2.4733	33.1124	20.4344	2144.200	4.0048	126.8905	28	9	13
MME2	0.1437	0.0524	2.8656	235.4195	1.1503	2.4098	27.5732	2565.5369	7.9658	147.1712	28	0	22
MDE2	0.1701	0.0810	3.4073	132.9046	6.6023	256.3632	22.2780	2902.5088	2.3905	154.9538	27	8	15
MLE	0.0916	0.0415	6.1998	234.3369	3.3455	102.3739	9.3112	1290.1463	2.6210	105.9846	38	5	7

EXAMPLE 3

Table 20 : n=50 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.1558	0.0720	3.9214	127.8468	4.9709	191.3257	21.1296	2895.3026	0.5705	108.8907	98	26	76
sum8	0.0566	0.0192	8.2676	229.9694	0.7040	3.4972	9.5364	1158.3459	0.9389	61.7961	95	52	53
sum10	0.0469	0.0172	9.0935	272.5700	1.4090	57.8996	1.6905	621.5959	0.2866	63.1468	93	58	49
sum5W	0.1366	0.0563	2.6891	84.2951	3.1477	82.5349	23.3876	2726.1660	1.8822	111.9643	126	14	60
sum8W	0.0947	0.0367	7.30341	264.2817	1.7908	21.9375	12.9921	1716.1256	1.2382	85.5882	126	30	44
sum10W	0.0870	0.0339	6.6871	184.8282	1.9281	32.3212	10.5096	1399.1677	0.7501	82.6224	111	47	42
MME2	0.0879	0.0251	4.3238	220.3678	1.1172	3.5327	20.1864	1704.7023	3.1858	69.6085	103	15	82
MDE2	0.1166	0.0516	8.4801	339.8992	3.3204	87.9873	10.2158	1859.6281	0.8651	86.0063	104	36	60
MLE	0.0622	0.0265	7.9243	261.8036	1.7782	52.9154	6.1997	900.5284	0.1803	118.0031	146	28	26

EXAMPLE 3

Table 21 : n=100 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.1082	0.0513	2.4145	53.5791	1.2374	7.4199	25.8900	2588.3289	3.0530	45.5359	35	6	9
sum8	0.0829	0.0341	4.7085	196.1577	0.7697	2.9687	18.3315	1941.7618	2.8653	45.8411	34	9	7
sum10	0.0695	0.0299	3.1309	64.1249	0.5104	1.9516	15.3783	1678.3733	2.5936	45.7369	31	12	7
sum5W	0.1008	0.0459	1.2463	42.0266	1.1767	8.6215	22.4993	2333.5672	2.5050	38.2381	42	3	5
sum8W	0.0921	0.0412	3.6062	127.0280	1.7276	42.7180	19.5725	2088.3626	3.1013	37.8173	38	8	4
sum10W	0.0893	0.0399	4.9742	237.2793	2.4355	128.4821	21.1195	2039.9787	2.9594	34.8655	41	6	3
MME2	0.0755	0.0203	2.7353	121.5071	0.8074	3.0538	21.9879	1743.4643	2.8534	25.5085	34	4	12
MDE2	0.1037	0.0465	5.5765	250.6321	1.0035	4.6942	23.5918	259.2999	3.0194	43.5107	34	9	7
MLE	0.0340	0.0089	3.0340	118.5588	0.4407	1.3072	8.0926	751.5368	1.6318	23.8664	43	4	3

EXAMPLE 3

Table 22 : n=100 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	-0.0594	0.0271	3.8816	90.6951	1.3872	47.2988	12.3396	1389.9993	0.73761	35.5174	134	21	45
sum8	0.0466	0.0237	7.0181	221.8305	0.9937	26.0793	9.1832	1190.5062	0.7550	44.1195	126	39	35
sum10	0.0497	0.0286	7.3457	233.8302	1.0493	23.4077	8.5886	1250.6887	0.7572	46.6672	124	45	31
sum5W	0.0629	0.0297	3.1809	86.7277	1.2537	26.9292	11.8208	1376.0600	0.5739	41.6052	166	10	24
sum8W	0.0583	0.0273	4.9877	154.7666	1.3492	27.9232	11.7644	1438.5523	1.1508	40.8754	160	24	16
sum10W	0.0566	0.0266	6.1581	211.7385	1.3117	43.0212	12.1084	1375.2329	1.2102	41.2386	159	29	12
MME2	0.0482	0.0114	3.3048	120.9269	0.5443	1.5045	12.8710	1034.7635	1.3995	31.4141	139	7	54
MDE2	0.0682	0.0358	9.0496	382.9288	1.5396	33.0417	12.6358	1741.5649	1.0993	45.2890	121	38	41
MLE	0.0166	0.0046	4.3811	112.7771	0.2933	0.5941	2.4341	387.9517	0.2826	27.4261	173	17	10

EXAMPLE 3

Table 23 : n=200 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0523	0.0263	2.0673	117.9591	0.8430	12.2976	15.5040	1358.8281	1.5780	25.1519	40	5	5
sum8	0.0303	0.0176	2.0263	37.9507	0.4316	4.4687	10.0298	892.5495	1.6186	25.9380	36	11	3
sum10	0.0493	0.0313	1.8937	37.6637	1.0429	19.4226	11.4389	1330.8497	1.5658	28.3247	36	10	4
sum5W	0.0411	0.0162	0.9614	40.4966	0.3304	0.8622	11.5283	956.9533	1.5137	19.4104	46	2	2
sum8W	0.0517	0.0287	2.1550	76.4782	0.7371	6.8346	12.7120	1245.7651	1.6795	25.2866	46	3	1
sum10W	0.0538	0.0300	1.3643	41.3618	0.7301	6.2226	13.1267	1316.7482	1.7822	25.4218	44	4	2
MME2	0.0208	0.0030	1.7295	80.8130	0.1893	0.2780	8.5976	502.0768	1.6637	17.0440	40	0	10
MDE2	0.0276	0.0180	3.4531	108.1846	1.3709	62.5578	8.5397	773.9930	0.9668	18.0386	37	10	3
MLE	0.0129	0.0026	3.0210	121.1394	0.1545	0.1884	4.6003	355.2242	1.0786	17.3306	46	3	1

EXAMPLE 3

Table 24 : n=200 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0408	0.0201	3.2006	100.3079	0.5428	5.4097	9.4674	1026.7791	0.5701	22.3505	161	18	21
sum8	0.0342	0.0224	3.5234	83.4923	1.0468	43.9262	7.8119	884.5337	0.6505	26.2081	148	37	15
sum10	0.0293	0.0214	5.0577	166.1816	1.1077	43.2501	6.6767	855.8202	0.4404	24.1499	155	33	12
sum5W	0.0292	0.0128	2.4208	51.2231	0.2889	0.9586	6.2829	640.2956	0.614	23.3370	183	7	10
sum8W	0.0280	0.0160	3.0929	79.6950	0.6960	18.4866	6.3909	661.5962	0.5984	22.0087	183	16	1
sum10W	0.0330	0.0191	2.6609	61.3827	0.6442	10.7806	7.4317	781.1703	0.8348	25.8887	176	20	4
MME2	0.0181	0.0026	2.6541	76.6325	0.1886	0.2173	5.1545	305.2499	0.8402	15.6371	154	3	43
MDE2	0.0299	0.0212	4.9722	148.1004	1.2650	51.4922	6.2214	775.5658	0.0877	19.4976	159	29	12
MLE	0.0073	0.0021	2.7711	63.0117	0.1244	0.1613	1.5242	177.4314	0.2537	14.0510	192	4	4

EXAMPLE 4

Table 25 : n=50 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0472	0.0248	3.3548	88.3167	1.4221	9.9403	1.2625	420.0243	0.1611	73.8257	40	5	5
sum8	- 0.0018	0.0120	6.8322	160.9880	0.8025	6.6447	- 1.7560	220.6138	0.6755	54.5458	39	9	2
sum10	0.0254	0.0194	4.9661	93.0968	1.1969	9.6289	5.5579	480.3537	0.8935	43.2072	34	11	5
sum5W	0.0473	0.0263	2.2623	45.8097	1.7778	22.7175	3.4995	644.9825	0.1090	57.5445	48	0	2
sum8W	0.0210	0.0135	3.9282	109.8147	1.2626	12.4094	0.6634	357.5244	0.3967	51.3855	43	5	2
sum10W	0.0210	0.0143	2.9307	46.9494	1.5962	31.7357	1.2005	413.7347	0.1055	49.3062	42	6	2
MME2	0.0377	0.0167	1.9343	25.0413	0.8803	3.9466	0.7632	303.9838	0.2366	50.1638	45	0	5
MDE2	0.0587	0.0289	2.6718	46.2064	2.8857	75.6175	3.6962	489.2358	1.0207	68.7397	36	5	9
MLE	0.0234	0.0109	2.5205	42.7821	0.9440	5.0678	0.5680	451.3896	0.2762	34.4834	45	4	1

EXAMPLE 4

Table 26 : n=50 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0375	0.0201	3.3665	78.1570	1.1491	7.7782	2.4376	481.5894	- 0.4290	55.0333	168	9	23
sum8	0.0029	0.0114	5.3936	115.7703	0.9809	18.3334	- 0.3958	303.8527	- 0.8701	49.6022	156	27	17
sum10	- 0.0024	0.0092	5.9717	185.8812	0.4788	3.4243	0.6746	366.1704	- 0.5625	49.0111	140	34	26
sum5W	0.0316	0.0187	2.8068	66.5991	1.0980	10.0361	3.4969	564.5748	0.3395	150.7101	187	4	9
sum8W	0.0133	0.0116	3.3888	68.8226	0.7672	6.1810	0.8538	359.5783	- 0.6207	46.4018	172	19	9
sum10W	0.0025	0.0099	4.3558	125.2110	0.7862	11.3581	0.2085	376.7806	- 0.7702	44.7142	171	23	6
MME2	0.0224	0.0130	2.5441	48.3496	0.9789	25.7455	1.3374	385.7507	0.1221	109.1529	182	0	18
MDE2	0.0376	0.0224	4.5729	149.2413	2.1068	52.5344	1.8440	429.9680	- 0.2295	59.2249	142	22	36
MLE	0.0150	0.0115	2.5500	38.1532	0.7762	5.1457	1.1972	406.1391	- 1.0280	180.8408	185	14	1

EXAMPLE 4

Table 27 : n=100 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0342	0.0164	1.3443	23.3108	0.6013	4.1717	- 2.4914	287.9376	- 0.0344	25.6753	46	1	3
sum8	0.0048	0.0063	2.8533	69.3925	0.1310	0.7186	- 3.4988	238.8526	- 0.1055	25.7044	44	6	0
sum10	- 0.0084	0.0038	2.3556	33.7246	0.0303	0.5814	- 3.3154	239.5368	.1210	27.6204	42	8	0
sum5W	0.0343	0.0148	0.9724	14.1215	1.0961	27.9307	0.6796	682.9488	- 0.3618	25.6253	49	0	1
sum8W	0.0203	0.0118	1.7097	29.7795	0.5611	7.3943	- 1.0580	553.3950	- 0.3786	26.5863	47	3	0
sum10W	0.0086	0.0096	1.7244	22.9261	0.4162	6.1027	- 2.6712	413.6974	- 0.2558	28.0372	44	6	0
MME2	0.0141	0.0047	0.7649	4.7659	0.2305	0.5256	- 4.1282	279.0685	- 0.4119	22.1480	49	1	0
MDE2	0.0366	0.0200	1.7632	24.6080	0.8400	7.6607	- 1.3642	261.3302	- 0.0666	26.0489	47	3	0
MLE	0.0102	0.0043	1.9395	46.1247	0.2490	0.6589	- 4.9542	256.3410	- 0.3267	19.9943	50	0	0

EXAMPLE 4

Table 28 : n=100, S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0201	0.0105	1.7993	31.1982	0.5168	2.5135	0.5538	283.2247	- 0.5835	28.1103	192	2	6
sum8	0.0100	0.0090	2.1458	42.9704	0.5397	7.6815	- 0.5309	259.3250	0.7400	28.1088	181	18	1
sum10	- 0.0047	0.0048	2.0279	27.3654	0.3001	2.9885	- 1.6422	166.0136	- 0.6273	27.4765	165	31	4
sum5W	0.0203	0.0099	0.8914	6.9955	0.6067	8.0176	0.8950	344.0266	- 0.6342	25.5423	199	0	1
sum8W	0.0156	0.0093	1.0557	10.2570	0.4856	3.4541	- 0.0893	299.5200	- 0.7997	25.6741	191	9	0
sum10W	0.0067	0.0066	1.1467	8.6234	0.3953	2.6563	- 1.5928	220.3724	- 0.7711	221.5431	181	19	0
MME2	0.0164	0.0070	0.9562	7.2783	0.4012	1.2283	- 0.5001	219.4333	- 0.7399	26.3636	196	0	4
MDE2	0.0365	0.0191	1.4728	19.7176	1.2920	31.4992	1.6045	306.1961	- 0.5369	29.6981	187	10	3
MLE	0.0130	0.0082	1.4627	40.6753	0.3677	1.3905	0.1287	235.9620	0.2289	100.1736	198	2	0

EXAMPLE 4

Table 29 : n=200 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0146	0.0064	0.2563	2.2485	0.2285	0.6171	- 0.4277	275.4013	0.1434	14.5104	50	0	0
sum8	0.0104	0.0114	0.4102	2.2191	0.4858	7.2772	0.8528	346.6067	- 0.0995	19.9704	47	3	0
sum10	0.0095	0.0113	0.4656	2.2067	0.4077	4.8804	1.3875	340.3257	- 0.1129	20.4518	45	5	0
sum5W	0.0096	0.0040	0.1437	1.0808	0.1695	0.3863	- 0.5723	233.7079	0.1591	12.1736	50	0	0
sum8W	0.0076	0.0039	0.1634	0.9849	0.1452	0.3456	- 0.3222	252.1953	0.1401	12.7849	49	1	0
sum10W	0.0014	0.0031	0.2333	0.9333	0.1156	0.3085	0.6639	239.3969	0.0594	13.0375	47	3	0
MME2	0.0062	0.0027	0.2339	1.2078	0.1270	0.2485	- 0.8218	199.2175	0.1495	11.8322	50	0	0
MDE2	0.0178	0.0131	0.3591	2.2195	0.4704	3.8271	2.1910	424.8615	- 0.0932	18.9319	48	2	0
MLE	0.0030	0.0024	0.2543	1.1967	0.1050	0.2205	0.0469	168.5917	0.2281	10.0354	50	0	0

EXAMPLE 4

Table 30 : n=200 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0121	0.0061	0.4243	3.2910	0.2049	0.5228	0.3481	213.7812	- 0.3725	15.0751	199	1	0
sum8	0.0126	0.0091	0.4708	2.0052	0.4516	9.8289	1.3054	260.2955	- 0.4970	16.4594	192	8	0
sum10	0.0047	0.0069	0.5553	2.0270	0.3179	5.1199	0.4752	212.8008	- 0.3351	15.2344	183	17	0
sum5W	0.0081	0.0037	0.2893	1.2391	0.1635	0.3593	- 0.6365	147.3133	0.2768	12.3917	200	0	0
sum8W	0.0076	0.0038	0.2988	1.1662	0.1556	0.3597	- 0.5843	148.8610	- 0.3023	12.5986	198	2	0
sum10W	0.0029	0.0030	0.3540	1.1354	0.1274	0.3027	- 0.4407	139.9159	- 0.2948	12.1188	192	8	0
MME2	0.0063	0.0033	0.3650	1.4472	0.1435	0.2782	- 0.5979	141.4150	- 0.3196	12.7225	200	0	0
MDE2	0.0178	0.0108	0.4331	2.0122	0.4959	7.0983	1.7371	278.3605	- 0.4199	15.9151	194	6	0
MLE	0.0045	0.0028	0.3596	1.4305	0.1479	0.2734	- 0.9211	128.4578	- 0.2529	11.2082	200	0	0

EXAMPLE 5

Table 31 : n=50 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0459	0.0327	2.8531	114.2841	10.5040	362.0204	3.7636	69.1793	10.8616	638.607	37	1	12
sum8	0.0180	0.0464	6.3513	289.1931	10.3416	274.3337	2.9530	80.2525	6.5925	753.3672	41	2	7
sum10	0.0125	0.0439	4.2637	109.4941	9.9774	293.9789	2.9749	82.8052	7.4696	733.1371	36	2	12
sum5W	0.0154	0.0383	4.8370	198.3455	8.9640	311.3449	3.4217	76.8719	8.6386	775.5544	43	0	7
sum8W	0.0269	0.0419	3.9502	133.3652	8.4942	206.7293	3.2861	79.8113	10.4127	862.2650	48	0	2
sum10W	0.0198	0.0398	3.2141	92.2833	8.0079	172.3392	3.0771	64.0435	8.1036	678.5588	45	2	3
MME2	0.0167	0.0329	0.7091	12.7854	6.0829	204.1643	3.5798	53.1902	9.3082	667.2663	44	0	6
MDE2	0.0037	0.0496	4.1905	94.5282	11.7446	443.1874	3.0213	74.4003	6.0733	750.0893	39	2	9
MLE	0.0365	0.0267	1.2738	24.9897	6.6308	163.7776	3.2628	53.2945	12.6336	728.002	47	0	3

EXAMPLE 5

Table 32 : n=50 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0096	0.0283	2.6402	63.4593	6.3727	203.7724	0.6023	53.3282	2.4903	551.3967	162	2	36
sum8	-0.0170	0.0445	6.5924	234.0999	6.7329	169.0972	0.9137	61.0402	-1.1320	702.8398	163	7	30
sum10	- 0.0343	0.0464	6.3848	174.8403	7.4708	253.6063	0.5224	60.1037	-3.2537	751.6172	153	9	38
sum5W	0.0029	0.0300	3.7349	130.0781	5.1996	144.5741	0.7310	49.8520	3.1762	599.6639	179	0	21
sum8W	- 0.0195	0.0420	6.5938	232.5677	5.0860	97.4433	0.7224	55.5183	1.1608	729.7102	188	4	8
sum10W	0.0142	0.0384	5.6513	185.5272	5.6884	144.9947	0.7218	50.2399	1.1716	655.5596	180	6	14
MME2	0.0247	0.0236	1.5006	34.0502	3.5438	86.3988	1.3296	44.0583	5.3662	508.8486	174	1	25
MDE2	- 0.0318	0.0475	7.2076	260.8481	7.1095	219.4344	0.9630	61.0066	- 1.6504	715.4304	145	14	41
MLE	0.0199	0.0242	2.4704	51.6435	4.5255	105.0818	1.0705	45.264	5.6218	557.9797	189	5	6

EXAMPLE 5

Table 33 : n=100 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0186	0.0190	0.5817	11.2371	3.3102	68.6026	1.7173	30.8392	4.1470	393.2442	46	0	4
sum8	- 0.0381	0.0494	4.3154	139.9217	3.7194	58.1572	0.6504	49.5837	- 1.9829	653.6001	47	3	0
sum10	- 0.064	0.0438	3.7226	87.5039	3.5292	58.8340	0.8512	47.3305	- 2.7580	669.9541	47	1	2
sum5W	0.0205	0.0188	0.6490	17.6634	3.7193	137.8422	1.5218	29.9326	4.6453	341.8183	49	0	1
sum8W	- 0.0039	0.0260	1.6838	42.3340	2.4768	29.2379	1.0511	31.6328	1.8523	410.9206	49	1	0
sum10W	- 0.0060	0.0262	2.1239	84.3553	2.6336	36.4865	1.1476	31.8363	1.0653	430.7565	50	0	0
MME2	- 0.0114	0.0145	1.2817	37.2259	1.1861	17.0812	0.5337	21.8167	0.1457	296.7172	50	0	0
MDE2	- 0.0246	0.0408	3.5736	104.9254	4.1716	78.9716	1.4252	33.3991	- 1.8855	582.3384	45	3	2
MLE	0.0053	0.0151	1.9536	107.7124	1.0422	7.1159	1.0205	26.3838	3.2247	290.6469	49	1	0

EXAMPLE 5

Table 34 : n=100 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0030	0.0246	1.8122	44.9910	2.7281	38.2949	0.5476	31.5710	0.2805	416.1373	189	0	11
sum8	- 0.0361	0.0444	4.4441	114.0024	3.2654	43.5954	0.4409	42.0638	- 3.3556	620.3352	185	8	7
sum10	- 0.0399	0.0459	4.7219	114.0848	3.2868	51.8833	0.3936	44.000	- 4.1588	646.4061	189	3	8
sum5W	0.0046	0.0209	1.4703	31.5856	2.4655	53.3791	0.5796	28.5277	1.2887	353.3410	196	2	2
sum8W	- 0.0196	0.0300	2.8605	64.3909	1.9647	18.6155	0.2632	33.6326	- 1.1275	465.9234	198	2	0
sum10W	- 0.0209	0.0314	3.1526	76.3167	2.0268	20.2947	0.2864	33.4193	- 1.3012	483.1321	199	1	0
MME2	- 0.0015	0.0156	1.8024	50.9474	1.0838	10.3858	0.2014	23.7426	0.3974	305.1643	194	0	6
MDE2	- 0.0328	0.0435	4.5452	137.6040	3.9372	79.0443	0.7304	39.3985	- 3.4712	619.0762	175	11	14
MLE	0.0029	0.0164	1.5984	39.4407	0.9521	5.0098	0.3392	271.1503	1.7876	296.6164	197	3	0

EXAMPLE 5

Table 35 : n=200 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0183	0.0196	0.3175	8.3427	1.9835	17.3026	1.4393	26.6085	4.3432	332.6441	50	0	0
sum8	0.0155	0.0245	0.7753	14.5324	2.1760	17.1184	1.6325	29.9186	3.6632	375.6339	50	0	0
sum10	0.0217	0.0262	1.2337	38.8971	2.6011	26.2448	1.7335	30.9901	4.3098	391.9854	48	2	0
sum5W	0.0133	0.0152	0.1630	5.1662	1.2394	7.6334	1.2525	23.4954	4.2338	282.7183	50	0	0
sum8W	0.0145	0.0160	0.3055	6.9259	1.4218	8.5995	1.4219	26.1987	4.4256	310.8010	50	0	0
sum10W	0.0096	0.0182	0.4549	8.0716	1.4023	10.0399	1.2869	25.5423	3.8536	312.5442	50	0	0
MME2	- 0.0056	0.0088	0.2579	4.0678	0.4206	1.4787	0.5677	14.6018	0.9364	186.6445	50	0	0
MDE2	0.0098	0.0265	1.6092	48.7554	2.5636	29.3396	1.5784	31.1128	3.4843	403.664	47	2	1
MLE	0.0106	0.0075	0.0367	2.4675	0.4793	1.4498	0.9191	18.1511	3.4782	181.0550	50	0	0

EXAMPLE 5

Table 36 : n=200 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0022	0.0183	0.7441	13.2722	1.4075	10.1965	0.4558	19.9437	1.1971	297.3353	200	0	0
sum8	- 0.0099	0.0266	1.6533	28.9451	1.8358	20.5947	0.5920	22.7887	- 0.03079	397.8495	199	0	1
sum10	- 0.0130	0.0296	2.3102	61.4444	2.1969	37.0971	0.5756	23.8918	- 0.6477	428.5658	196	4	0
sum5W	0.0017	0.0144	0.5876	8.3320	0.9109	5.0977	0.4491	17.3182	1.6642	241.6172	200	0	0
sum8W	0.0023	0.0173	0.8725	13.2859	1.0897	5.6610	0.6287	18.9497	1.9419	282.0317	199	1	0
sum10W	- 0.0024	0.0184	0.9963	14.9079	1.220	6.9400	0.5571	19.2008	1.6445	288.4384	199	1	0
MME2	- 0.0101	0.0104	0.9018	17.7886	0.4259	1.9784	0.0884	14.8280	- 0.1376	199.8895	198	0	2
MDE2	- 0.0144	0.0277	2.4088	75.9840	1.8922	20.2609	0.6280	22.9739	- 0.4763	405.0916	182	11	7
MLE	- 0.0032	0.0104	0.6950	8.9050	0.4306	2.5395	0.2930	14.5016	1.1647	184.2743	200	0	0

EXAMPLE 6

Table 37 : n=50 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	- 0.0844	0.1166	11.0831	487.4909	8.6972	265.3806	0.9107	564.8771	8.4733	826.5703	20	2	28
sum8	0.0527	0.0840	5.2571	113.4300	11.8216	311.6562	3.3723	265.6373	40.6161	9347.0108	16	13	21
sum10	0.0561	0.1029	7.2790	351.5632	10.7314	280.6166	1.2220	302.9171	20.5681	9250.9111	14	10	26
sum5W	- 0.508	0.1037	6.8144	165.6477	8.4495	342.3365	- 0.8496	820.0686	13.0875	1304.6988	26	1	23
sum8W	0.0304	0.0879	0.6769	302.3624	9.6888	269.4467	1.8913	654.8785	17.9574	1199.1989	23	3	24
sum10W													
MME2	0.0342	0.0897	3.5910	67.0157	4.3717	93.1440	- 5.8707	783.4325	30.2286	2167.4275	18	0	32
MDE2	- 0.0222	0.1099	12.4416	584.9376	8.0248	186.0564	1.9513	470.1958	11.3396	1594.6234	23	8	19
MLE	0.0441	0.1015	6.5401	326.5487	9.0001	250.7304	- 1.4806	769.0671	24.5932	2165.8961	41	2	7

EXAMPLE 6

Table 38 : n=50 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	-0.0530	0.0973	7.2341	283.1365	5.9840	158.4596	-2.6364	639.8200	8.4062	1071.5863	81	10	109
sum8	-0.0383	0.1003	6.6390	156.9424	9.1753	322.4797	-2.7326	596.9629	13.2474	2678.8610	73	51	76
sum10													
sum5W	-0.0295	0.0902	4.1429	92.8768	5.4588	181.5233	-4.6494	716.5275	10.5115	1317.8181	101	9	90
sum8W	0.0072	0.0988	7.1894	300.3433	9.6879	328.7160	-1.9753	670.2258	11.8549	1139.4961	99	32	69
sum10W													
MME2	-0.0076	0.0818	2.0844	25.1447	2.8758	38.3347	-6.5206	753.7334	13.8061	1417.2499	98	2	100
MDE2	-0.0169	0.1009	8.1535	328.2757	10.3700	413.5040	0.6695	605.0022	6.6190	1189.8803	97	28	75
MLE	0.0082	0.0917	4.6963	177.6870	6.4629	187.7420	-4.3904	742.1558	18.7673	1954.5357	147	21	32

EXAMPLE 6

Table 39 : n=100 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	- 0.0441	0.1202	5.6666	297.5640	6.8332	182.9657	- 0.1154	822.4320	3.6260	807.3119	20	0	30
sum8	- 0.0018	0.1307	6.4131	204.0925	12.1007	510.3995	3.2306	963.9573	0.8268	614.6507	21	11	18
sum10													
sum5W	- 0.0108	0.1169	4.3077	114.7003	5.2986	97.4254	- 2.6879	821.3692	8.8826	1079.9960	26	1	23
sum8W	- 0.0646	0.1376	13.2827	678.9027	7.9925	292.1037	1.5790	1200.6260	- 2.7549	896.6345	24	10	16
sum10W													
MME2	- 0.1330	0.1255	3.3735	41.5629	2.6553	47.4762	- 13.6987	1007.9864	1.0869	666.5182	27	2	21
MDE2	0.0705	0.1165	2.7050	42.8294	13.0431	486.0807	5.7954	605.7238	4.6711	784.4147	25	6	19
MLE	- 0.0614	0.1072	3.6377	79.7522	3.6201	57.4736	- 10.2728	901.9951	11.3300	1525.4230	36	4	10

EXAMPLE 6

Table 40 : n=100 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0012	0.1066	3.8827	127.3625	7.6009	247.2068	- 0.9495	724.3696	3.2790	648.1954	79	13	108
sum8	- 0.0572	0.1276	9.1352	315.0832	7.5536	277.2545	- 3.0149	801.3917	0.5038	666.5918	78	61	61
sum10													
sum5W	- 0.0212	0.1077	4.1043	95.8341	3.8540	62.6207	- 5.1242	837.2595	5.4555	8881.6931	108	10	82
sum8W	- 0.0476	0.1201	7.4284	288.1942	6.9865	259.9928	- 3.5997	873.1360	3.9556	814.1733	102	42	56
sum10W													
MME2	- 0.0748	0.0964	2.6561	41.1071	1.7864	21.4038	- 9.7368	864.8142	6.0339	986.7126	109	4	87
MDE2	- 0.0179	0.1168	7.9642	315.1022	9.6105	370.2946	- 0.0522	739.6416	1.5357	600.5308	99	27	74
MLE	- 0.0467	0.1029	4.5941	131.3282	4.4406	132.2992	- 7.5500	859.2831	9.6258	1279.9063	145	13	42

EXAMPLE 6

Table 41 : n=200 , S=50

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	0.0707	0.1039	3.2582	84.3016	5.7573	140.2115	- 0.1772	675.5737	10.3265	777.9740	22	6	22
sum8	0.0169	0.1332	5.8650	170.5249	13.4887	661.7887	- 1.2081	845.4837	9.7812	774.8609	25	11	14
sum10													
sum5W	- 0.0180	0.1229	2.4849	31.5165	2.1173	13.8741	- 9.4084	1004.0968	6.6756	943.2763	30	6	14
sum8W	0.0092	0.1232	6.2666	292.8194	5.6439	249.3302	- 6.9457	914.2203	8.5740	762.2148	31	6	13
sum10W													
MME2	0.0653	0.1049	2.3043	43.2970	2.8194	19.7630	- 1.6342	718.9607	13.6667	917.2351	30	4	16
MDE2	0.0406	0.1063	4.7178	229.1258	5.9452	136.7825	- 2.5759	661.2189	9.2700	785.8112	25	7	18
MLE	- 0.0293	0.1090	4.0475	125.6704	2.9887	71.0907	- 9.3358	812.1315	6.9901	904.6175	35	5	10

EXAMPLE 6

Table 42 : n=200 , S=200

	P		κ_1		κ_2		μ_1		μ_2				
	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE	C1	C2	C3
sum5	- 0.335	0.1189	4.3249	100.9884	4.1620	116.8732	- 5.8302	852.2250	4.0273	719.5880	96	19	85
sum8	- 0.0640	0.1501	8.5856	281.4976	7.7222	311.5973	- 6.1226	977.1085	0.3145	588.2998	86	63	51
sum10													
sum5W	- 0.0343	0.1221	4.2019	149.8818	2.9457	60.5601	- 7.4476	926.7553	6.3894	1043.3179	123	19	58
sum8W	- 0.0875	0.1346	6.6611	232.3199	3.2972	97.4497	- 12.1824	1035.7351	3.5490	716.3907	125	31	44
sum10W													
MME2	- 0.0640	0.1146	2.1363	29.2963	1.8170	30.7562	- 12.5331	1089.9155	5.5869	796.1940	112	14	74
MDE2	- 0.0498	0.1316	5.2700	152.7223	6.8490	254.7611	- 5.9703	861.6006	3.8170	738.1572	100	41	59
MLE	- 0.0569	0.1279	5.2645	191.5759	5.2210	181.2644	- 8.7522	1018.9418	4.8201	936.8347	152	15	33

4.2 Real data

Table 43 shows the final estimates for the different methods when applied to the Turtle data set. Mardia(1975a) has analysed this data set using a program of Jones and James(1969) to obtain the MLE's and quoted results obtained from fitting a circular histospline (see Boneva et. al 1971).

Mardia's MLE values are identical to ours and the estimates obtained from the circular histospline are also shown in table 43. All the methods give similar values for the final estimates except for κ_2 which ranges from 4.81 to 10.47. MME2 is again the quickest and there is little to choose between MLE and MDE2. An interesting point to note about this data set is that the assumption of two modes 180° apart seems reasonable but that of equal concentrations seems most unlikely. However , the estimates of p , μ_1 and μ_2 are similar to those obtained using the three parameter model (2.5).

Table 43

	P	κ_1	κ_2	μ_1	μ_2	CPU time (sec.)
sum5	0.83	2.92	10.20	63.59	240.45	0.63
sum8	0.83	2.92	8.70	63.54	240.87	0.72
sum10	0.83	2.93	8.54	63.54	241.21	0.92
sum5W	0.83	2.88	8.14	63.36	240.74	0.65
sum8W	0.83	2.89	7.46	63.35	240.81	0.80
sum10W	0.83	2.89	7.42	63.36	240.87	0.85
MME1	0.84	2.67	10.47	63.24	240.42	64.80
MME2	0.82	2.91	4.81	63.15	240.19	0.42
MDE1	0.82	2.97	6.21	63.19	240.14	17.79
MDE2	0.83	2.93	8.64	63.54	241.18	2.79
MLE	0.84	2.62	8.45	63.47	241.20	3.53
Circular Histospline	0.85	1.94	7.76	65.30	239.00	----

Final estimates using methods described in Section 3 for Turtle data (n=76 observations).

CHAPTER 5

SIMULATING SPHERICAL DISTRIBUTIONS

5.1 The Fisher Distribution

A spherical random variable (θ , ϕ) is said to have a Fisher distribution if its probability density function is given by

$$f((\theta,\phi);(\alpha,\beta),\kappa) = C_F \exp[\kappa(\sin\theta \sin\alpha \cos(\phi-\beta) + \cos\theta \cos\alpha)] \sin\theta ,$$

$$0 \le \theta , \alpha \le \pi , 0 \le \phi , \beta \le 2\pi , \kappa > 0 \qquad (5.1)$$

where $C_F = \kappa / (4 \pi \sinh \kappa)$. (5.2)

(α,β) are the polar coordinates of the mean direction and κ is a measure of concentration about this direction.

The distribution is unimodal and is rotationally symmetric about (α,β). For large values of κ the distribution is tightly clustered about (α,β). For $\kappa = 0$, (5.1) reduces to the uniform distribution on the sphere i.e.

$$f(\theta,\phi) = \sin\theta / (4 \pi).$$

If $\alpha = 0$ in (5.1) , the pdf reduces to

$$f((\theta,\phi);(0,\beta),\kappa) = C_F \exp(\kappa \cos\theta) \sin\theta , \ 0 \le \theta \le \pi , 0 \le \phi < 2\pi \qquad (5.3)$$

so that the longitude ϕ is distributed uniformly on $(0,2\pi)$, independently of the colatitude θ and β is arbitrary.

The distribution (5.1) was introduced by Arnold(1941) and studied extensively by Fisher(1953) and is the basic model for directions in three dimensions. For further details see Fisher, Lewis and Embleton(1987) and Mardia(1972).

5.2 Simulation of Fisher Distribution

We use the method described by Fisher , Lewis and Willcox(1981) to simulate random samples from the Fisher distribution with mean direction (α,β) and κ specified. The procedure is as follows :

(1) Take u_1, u_2 to be pseudo-random observations from U(0,1).

(2) Set $\lambda = \exp(-2\kappa)$

(3) Colatitude $\theta = 2 \sin^{-1}[-\{\ln(u_1(1-\lambda) + \lambda)\} / 2\kappa]^{1/2}$

(4) Longitude $\phi = 2\pi u_2$

(5) Rotate (θ,ϕ) to (θ',ϕ') as follows :

$$\begin{pmatrix} \sin\theta' \cos\phi' \\ \sin\theta' \sin\phi' \\ \cos\theta' \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta & -\sin\beta & \sin\alpha\cos\beta \\ \cos\alpha\sin\beta & \cos\beta & \sin\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \qquad (5.4)$$

(6) (θ',ϕ') is the required pseudo-random variate.

(7) To generate a random sample of size n steps 2 - 6 are repeated n times with new observations u_1 and u_2 each time.

Fisher, Lewis and Willcox (1981) make the point that " This method of generating (θ,ϕ) avoids the rounding errors inherent in Mardia's procedure(Mardia , 1972 , p 232) for large κ."

5.3 Goodness of Fit tests for the Fisher Distribution

We use two methods described by Fisher, Lewis and Embleton(1987) to test whether the simulated samples adequately fit a Fisher distribution. The first method is based on graphical displays which show the general behaviour of the model. The second method deals with formal significance tests. Both methods are valid if $\kappa \geq 3$ (see Fisher & Best (1984)).

For the graphical displays we use two kinds of plots : colatitude plots and longitude plots. Also, for the formal tests we have a colatitude test and a longitude test.

To describe each kind of plot we do the following :

(1) Calculate the sample mean direction ($\hat{\alpha}_o$, $\hat{\beta}_o$) by solving the following equations :

$$\cos \hat{\alpha}_o = N / R , \qquad \sin \hat{\beta}_o = (M / R) \operatorname{cosec} \hat{\alpha}_o ,$$

$$\cos \hat{\beta}_o = (L / R) \operatorname{cosec} \hat{\alpha}_o \tag{5.5}$$

$$\text{where } L = \sum_{i=1}^{n} \sin\theta_i \cos\phi_i , \; M = \sum_{i=1}^{n} \sin\theta_i \sin\phi_i , \; N = \sum_{i=1}^{n} \cos\theta_i \tag{5.6}$$

are the direction cosines and

$$R = (L^2 + M^2 + N^2)^{1/2} \tag{5.7}$$

is the sample resultant length.

(2) Rotate (θ_i,ϕ_i) , $i = 1,.......,n$ to the pole ($\hat{\alpha}_o$, $\hat{\beta}_o$) to obtain (θ_i' , ϕ_i') by using

$$\begin{pmatrix} \sin\theta_i' \cos\phi_i' \\ \sin\theta_i' \sin\phi_i' \\ \cos\theta_i' \end{pmatrix} = \begin{pmatrix} \cos\hat{\alpha}_o \cos\hat{\beta}_o & \cos\hat{\alpha}_o \sin\hat{\beta}_o & -\sin\hat{\alpha}_o \\ -\sin\hat{\beta}_o & \cos\hat{\beta}_o & 0 \\ \sin\hat{\alpha}_o \cos\hat{\beta}_o & \sin\hat{\alpha}_o \sin\hat{\beta}_o & \cos\hat{\alpha}_o \end{pmatrix} \begin{pmatrix} \sin\theta_i \cos\phi_i \\ \sin\theta_i \sin\phi_i \\ \cos\theta_i \end{pmatrix} \tag{5.8}$$

(3) For the colatitude plot we do the following :

(i) Set $X_i = 1 - \cos\theta_i'$, $i = 1,\dots\dots,n$ (5.9)

(ii) Re-arrange the X_i's in increasing order i.e. $X_{(1)} < X_{(2)} \dots\dots < X_{(n)}$.

(iii) Display the quantile-quantile (Q-Q) plot by plotting the points $(a_i, X_{(i)})$, $i = 1,\dots\dots,n$ where the numbers $a_1, a_2,\dots\dots\dots a_n$ are given by

$a_i = F^{-1}((i-\frac{1}{2})/n)$, and F^{-1} is the inverse of the unit exponential distribution i.e. $F^{-1}((i-\frac{1}{2})/n) = -\log(1 - ((i-\frac{1}{2})/n))$.

The plot should be approximately linear, passing through the origin with slope giving an estimate of $1/\kappa$.

(4) For the longitude plot we do the following

(i) Set $X_i = \phi_i' / 2\pi$, $i = 1,\dots\dots\dots,n$. (5.10)

(ii) Re-arrange the X_i's in increasing order i.e. $X_{(1)} < X_{(2)} \dots\dots < X_{(n)}$.

(iii) Display the quantile-quantile (Q-Q) plot by plotting the points $(b_i, X_{(i)})$, $i = 1,\dots\dots,n$ where the numbers $b_1, b_2,\dots\dots\dots b_n$ are given by

$b_i = F^{-1}((i-\frac{1}{2})/n)$, and F^{-1} is the inverse of the uniform distribution i.e.

$$F^{-1}((i-\frac{1}{2})/n) = (i-\frac{1}{2})/n.$$

The plot should be approximately linear, passing through the origin with slope of 45°.

The formal tests are described as follows :

(1) For the colatitude test

(i) Calculate the Kolmogorov-Smirnov statistic D_n where

$$D_n = \text{maximum}\ (D_n^+, D_n^-), \tag{5.11}$$

with $D_n^+ = \text{maximum of } [(i/n) - F(X_{(i)})] \quad , i = 1, \ldots\ldots n$ (5.12)

and $D_n^- = \text{maximum of } [F(X_{(i)}) - (i-1)/n] \quad , i = 1, \ldots\ldots n.$ (5.13)

Here X_i is given by (5.9) and $F(X_{(i)}) = 1 - \exp(-\hat{\kappa}\, X_{(i)})$ with

$$\hat{\kappa} = (n-1)/\sum_{i=1}^{n} X_i.$$

(ii) Calculate the test statistic

$$M_E(D_n) = (D_n - 0.2/n)(\sqrt{n} + 0.26 + (0.5/\sqrt{n})).$$

We reject the hypothesis of a Fisher distribution if $M_E(D_n)$ is too large.

Critical values of $M_E(D_n)$ are given in Appendix A8 of Fisher, Lewis and Embleton (1987).

(2) For the longitude test

(i) Calculate the Kuiper statistic V_n where

$V_n = D_n^+ + D_n^-$, with D_n^+ and D_n^- given by (5.12) and (5.13) respectively.

Here X_i is given by (5.10) and $F(X_{(i)}) = X_{(i)}$.

(ii) Calculate the test statistic

$$M_U(V_n) = V_n \left(\sqrt{n} - 0.567 + (1.623/\sqrt{n}) \right)$$

We reject the hypothesis of a Fisher distribution if $M_U(V_n)$ is too large.

Critical values of $M_U(V_n)$ are again given in Appendix A8 of Fisher, Lewis and Embleton(1987).

To test the goodness of fit we simulated three different samples of size n = 50 from two different Fisher distributions using the method described in Section (5.2). The parameters of the simulated distributions were

(i) $\kappa = 6.0$, $\alpha = 30$ and $\beta = 120$,

(ii) $\kappa = 9.0$, $\alpha = 100$ and $\beta = 300$.

Hence we have three samples from the Fisher distribution with parameters given by (i) and three samples from the distribution (ii). For each of these six samples the graphical and formal tests discussed earlier were carried out.

The results obtained are as follows :

Fisher distribution (i)

Sample 1 : The colatitude plot is shown in Figure (10) with slope = 0.151 and estimated $\kappa = 6.6$. The longitude plot is shown in Figure (11).

The colatitude test is

$M_E(D_n) = 0.7477 < 0.990$, the upper 10% point.

The longitude test is

$M_U(V_n) = 1.069 < 1.138$, the upper 10% point.

Both the plots and the formal tests show that the fit to a Fisher distribution is adequate.

Sample 2 : The colatitude plot is shown in Figure (12) with slope = 0.135 and estimated $\kappa = 7.4$. The longitude plot is shown in Figure (13).

The colatitude test is

$M_E(D_n) = 0.8389 < 0.990.$

The longitude test is

$M_U(V_n) = 0.8887 < 1.138.$

Both the plots and the formal tests show that the fit to a Fisher distribution is adequate.

Sample 3 : The colatitude plot is shown in Figure (14) with slope = 0.177 and estimated κ = 5.65. The longitude plot is shown in Figure (15).

The colatitude test is

$M_E(D_n) = 0.5674 < 0.990.$

The longitude test is

$M_U(V_n) = 0.7732 < 1.138.$

Both the plots and the formal tests show that the fit to a Fisher distribution is adequate.

Fisher distribution (ii)

Sample 1 : The colatitude plot is shown in Figure (16) with slope = 0.1 and estimated κ = 10.0. The longitude plot is shown in Figure (17).

The colatitude test is

$M_E(D_n) = 0.767 < 0.990.$

The longitude test is

$M_U(V_n) = 1.0628 < 1.138.$

Both the plots and the formal tests show that the fit to a Fisher distribution is adequate.

Sample 2 : The colatitude plot is shown in Figure (18) with slope = 0.0897 and estimated κ = 11.148. The longitude plot is shown in Figure (19).

The colatitude test is

$M_E(D_n) = 0.8695 < 0.990.$

The longitude test is

$M_U(V_n) = 0.8980 < 1.138.$

Both the plots and the formal tests show that the fit to a Fisher distribution is adequate.

Sample 3 : The colatitude plot is shown in Figure (20) with slope = 0.118 and estimated κ = 8.47. The longitude plot is shown in Figure (21).

The colatitude test is

$M_E(D_n) = 0.5634 < 0.990.$

The longitude test is

$M_U(V_n) = 0.7716 < 1.138.$

Both the plots and the formal tests show that the fit to a Fisher distribution is adequate.

As to be expected, there is some variability between the colatitude and longitude plots for each sample. However, none of the plots show major deviations from those expected under a Fisher model. These conclusions are confirmed by the formal tests where the goodness of fit to the Fisher model is accepted as satisfactory for all six simulated samples.

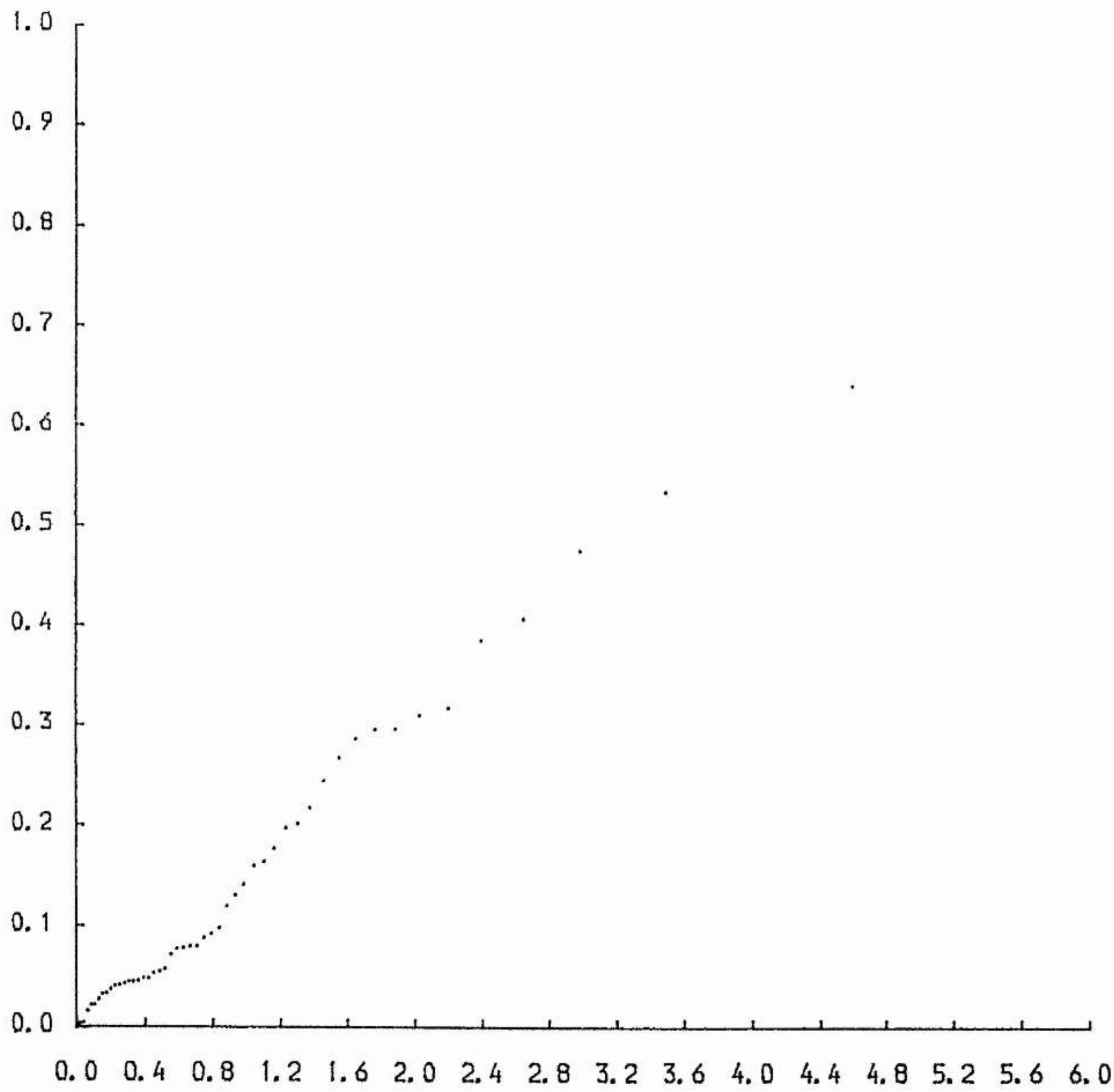


Figure (10)
The colatitude plot for sample 1 of Fisher distribution (i).

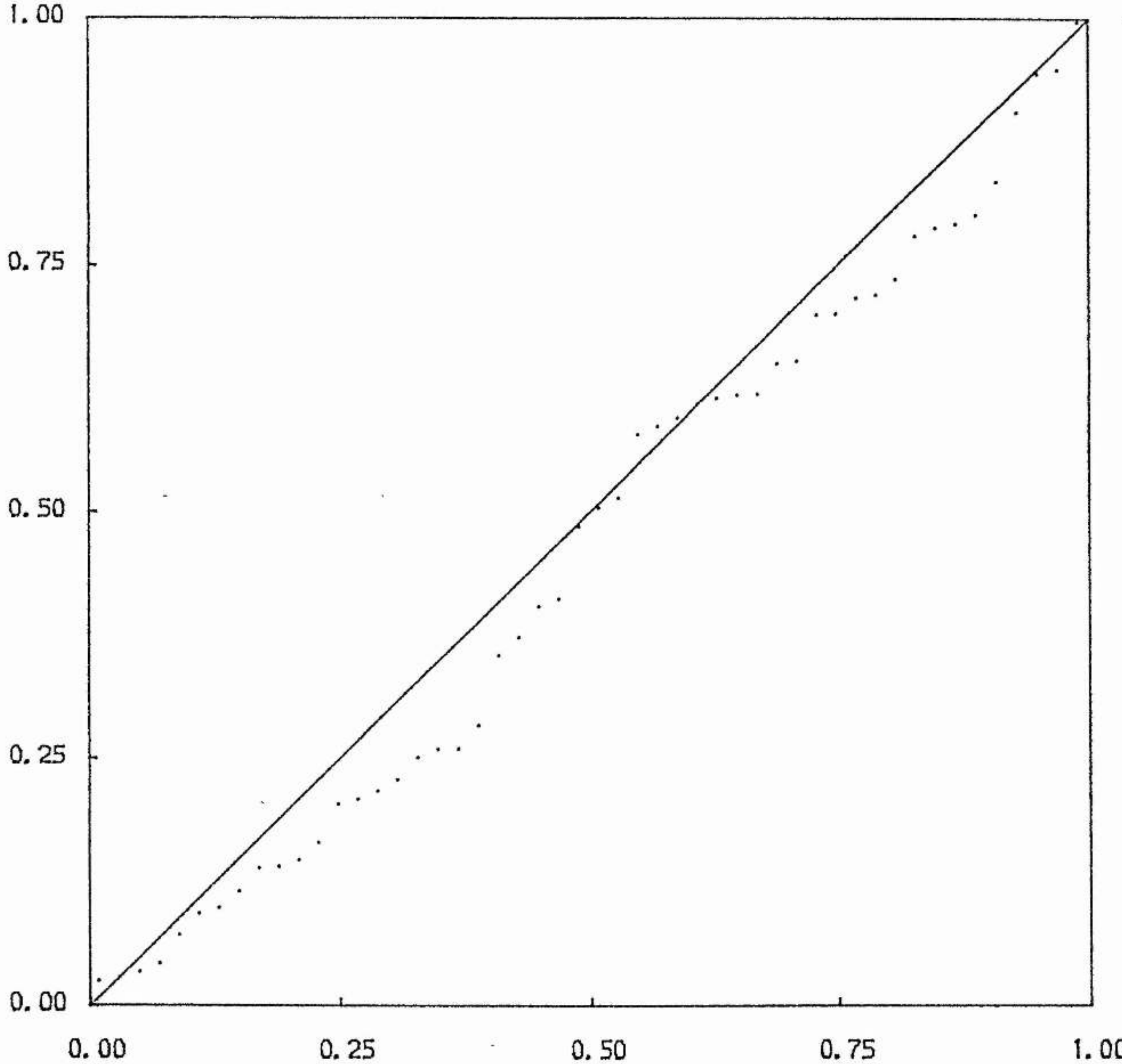


Figure (11)
The longitude plot for sample 1 of Fisher distribution (i).

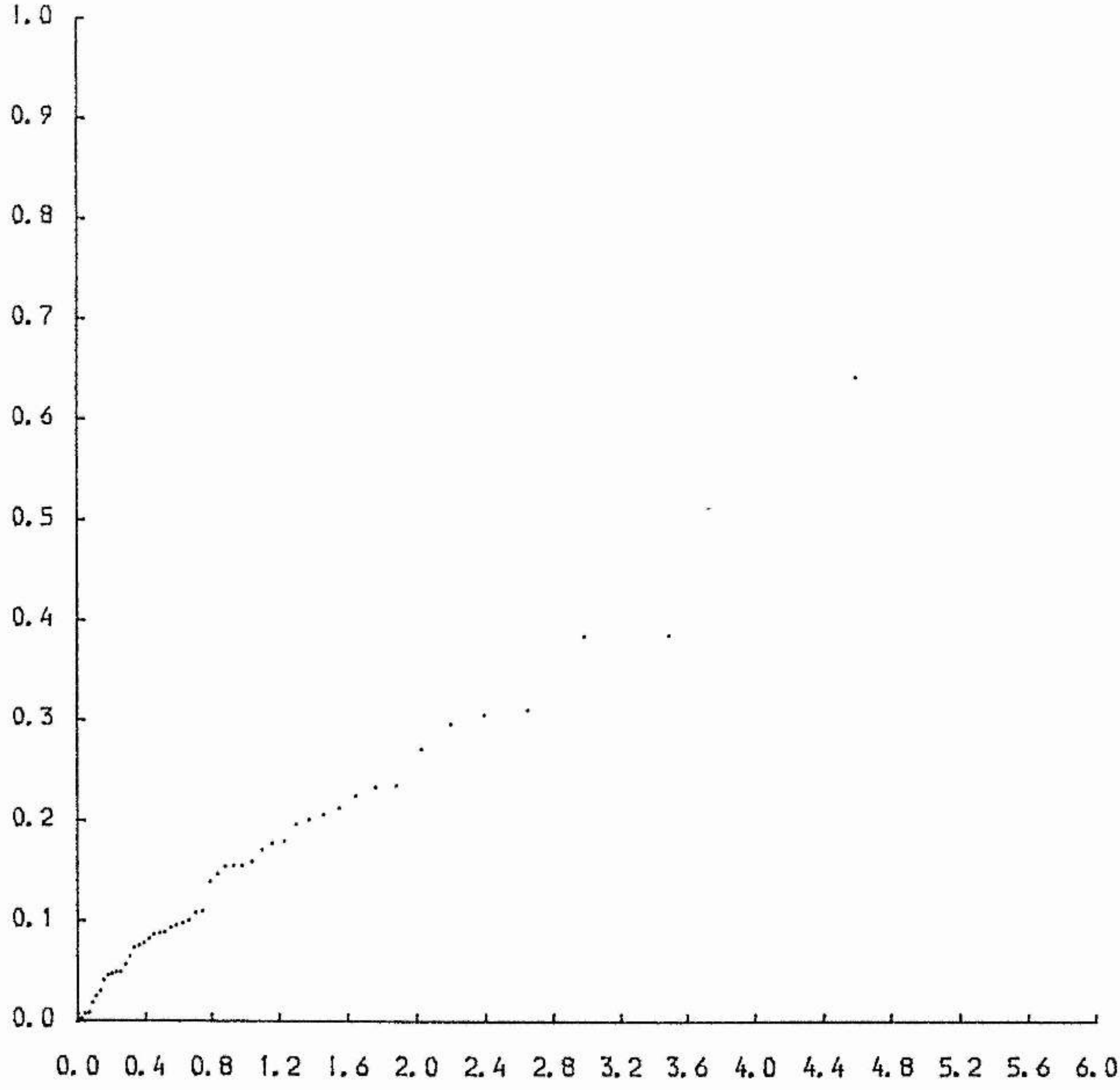


Figure (12)
The colatitude plot for sample 2 of Fisher distribution (i).

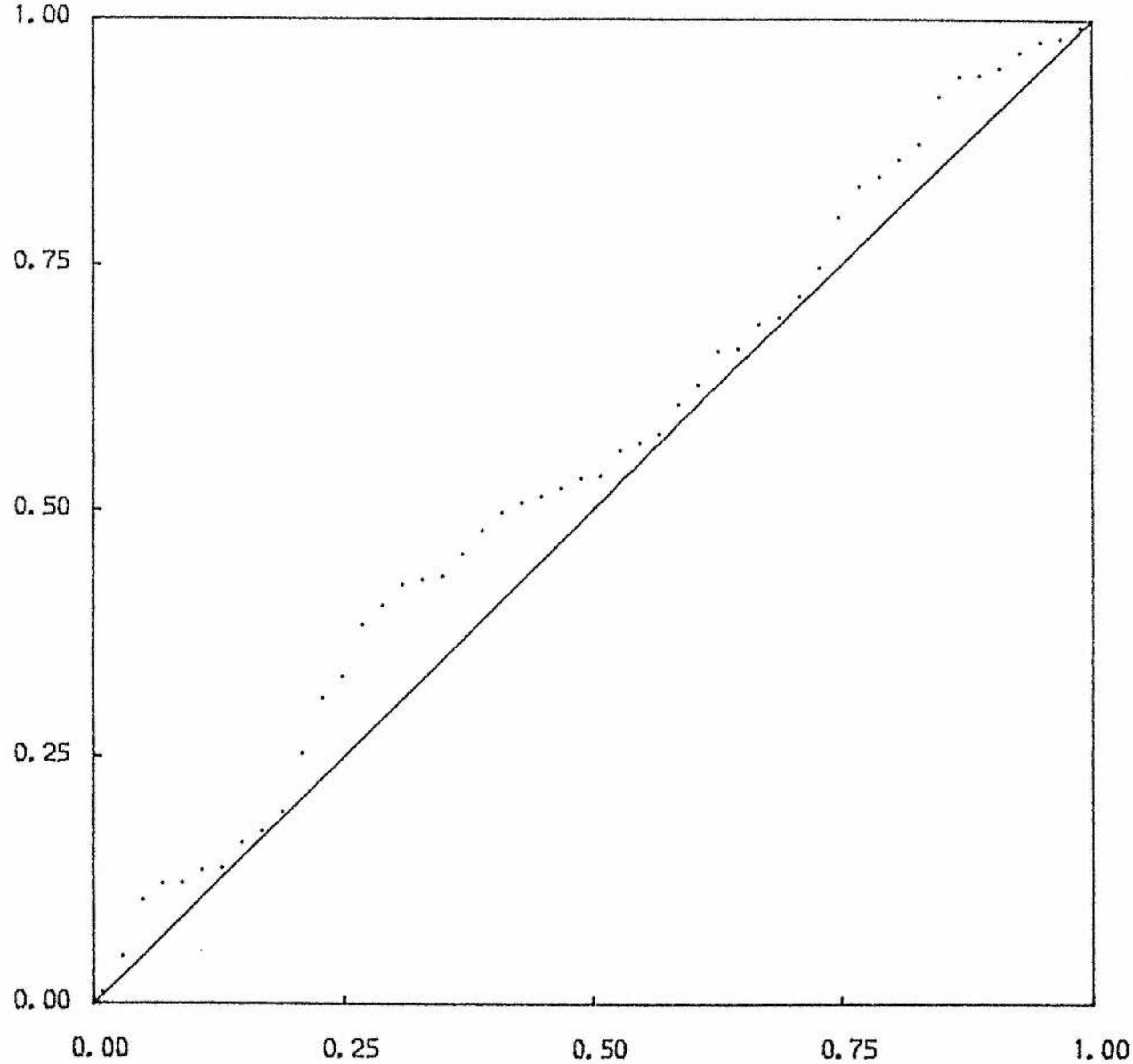


Figure (13)
The longitude plot for sample 2 of Fisher distribution (i).

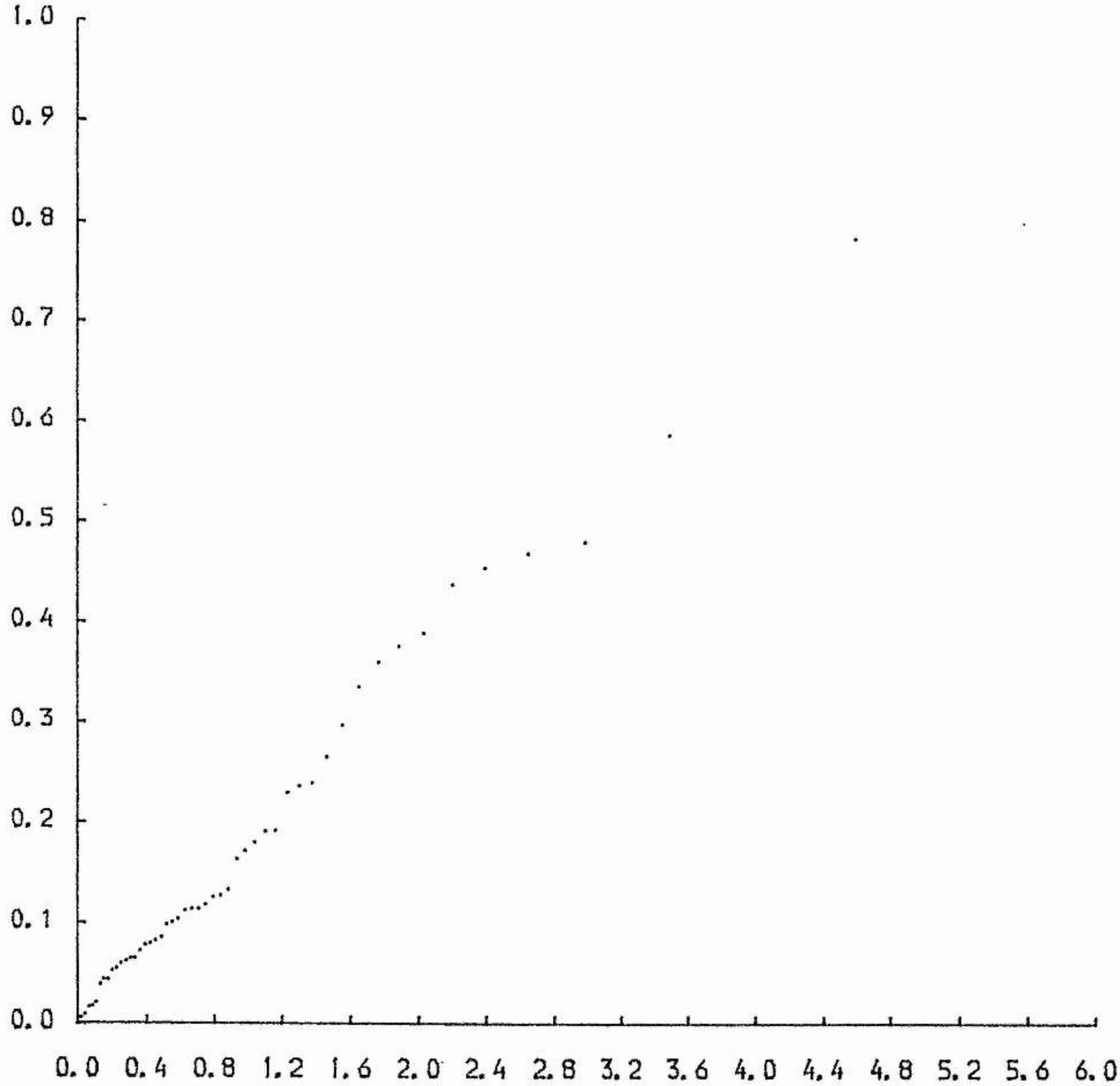


Figure (14)
The colatitude plot for sample 3 of Fisher distribution (i).

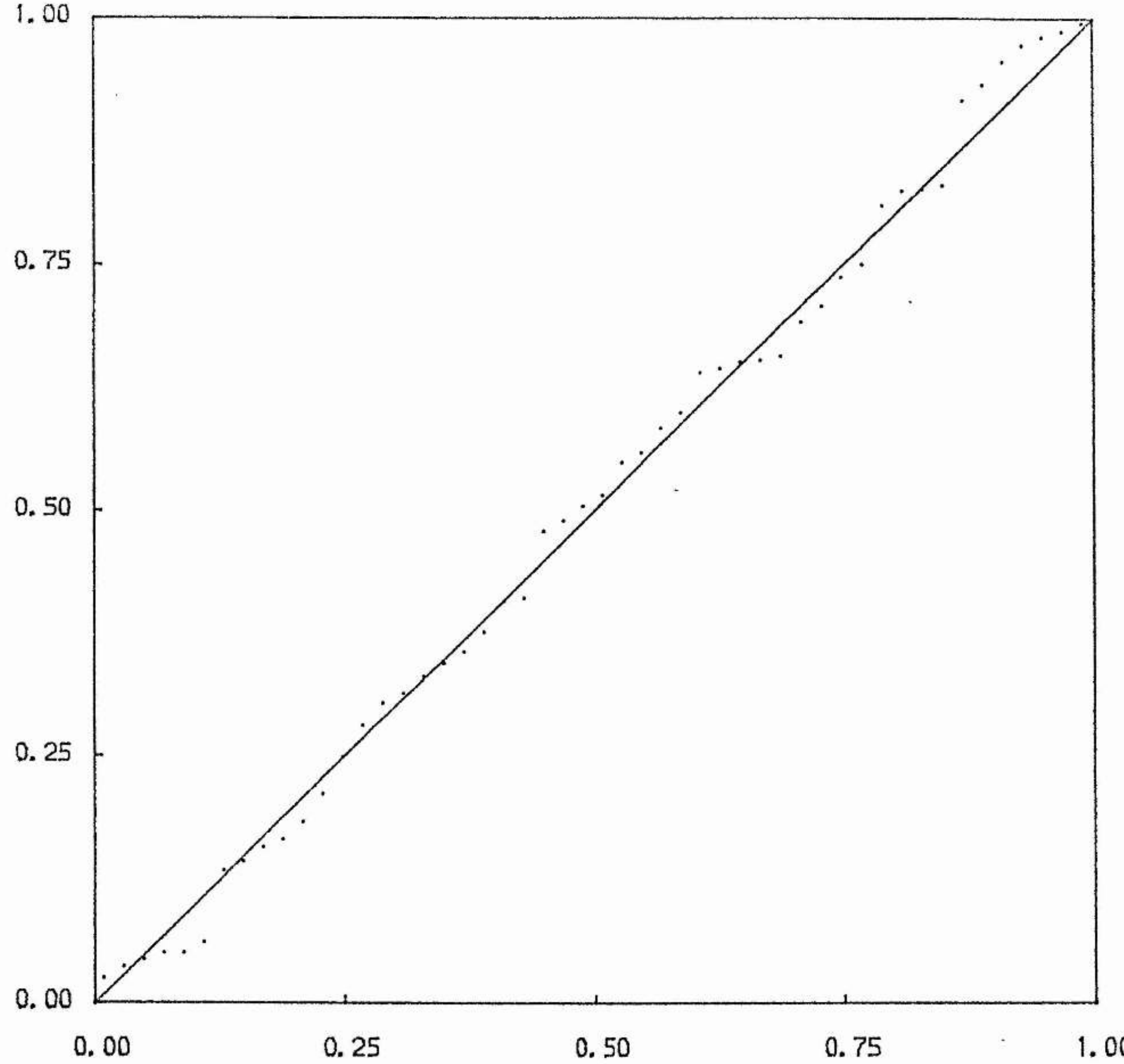


Figure (15)
The longitude plot for sample 3 of Fisher distribution (i).

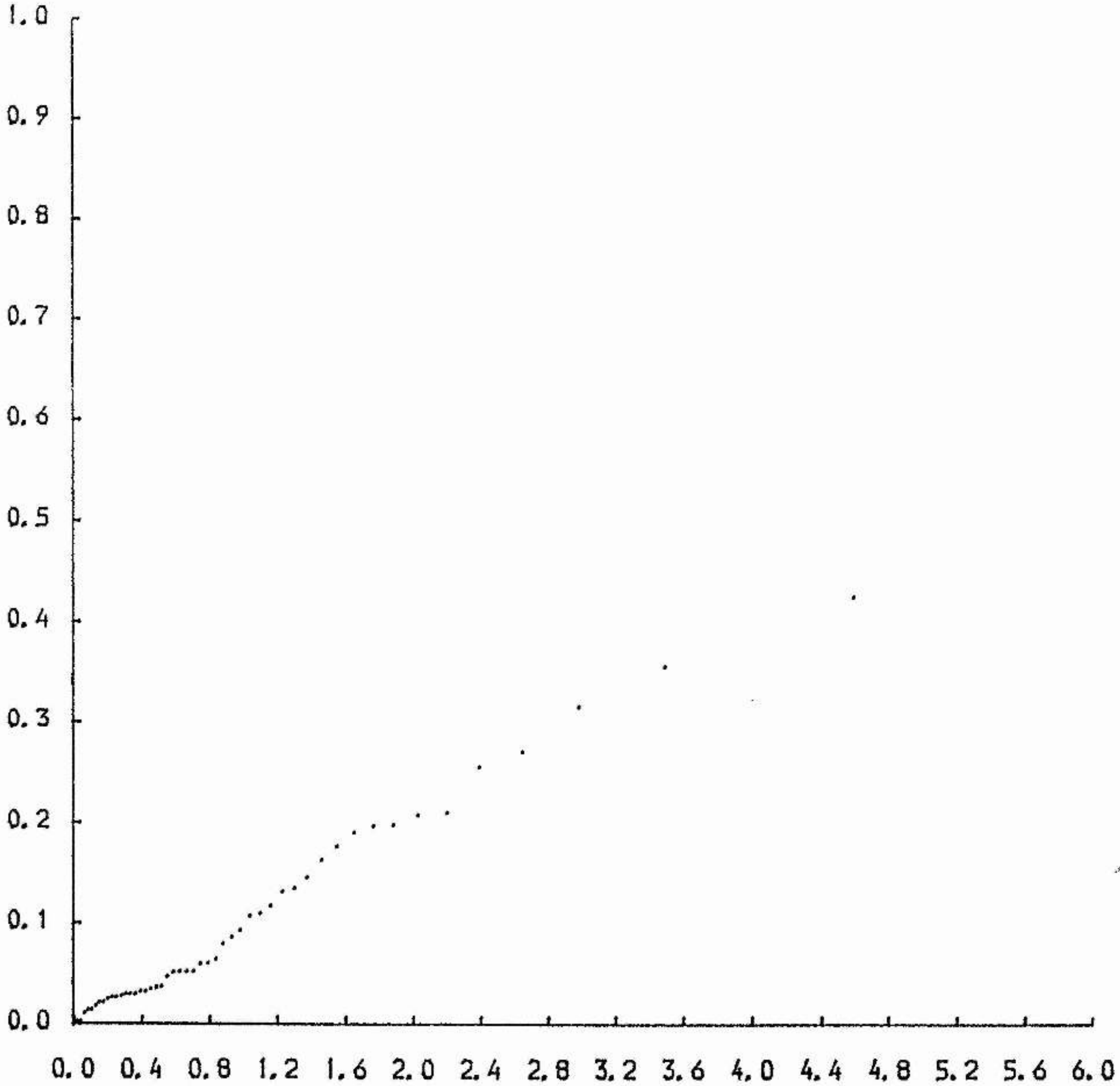


Figure (16)
The colatitude plot for sample 1 of Fisher distribution (ii).

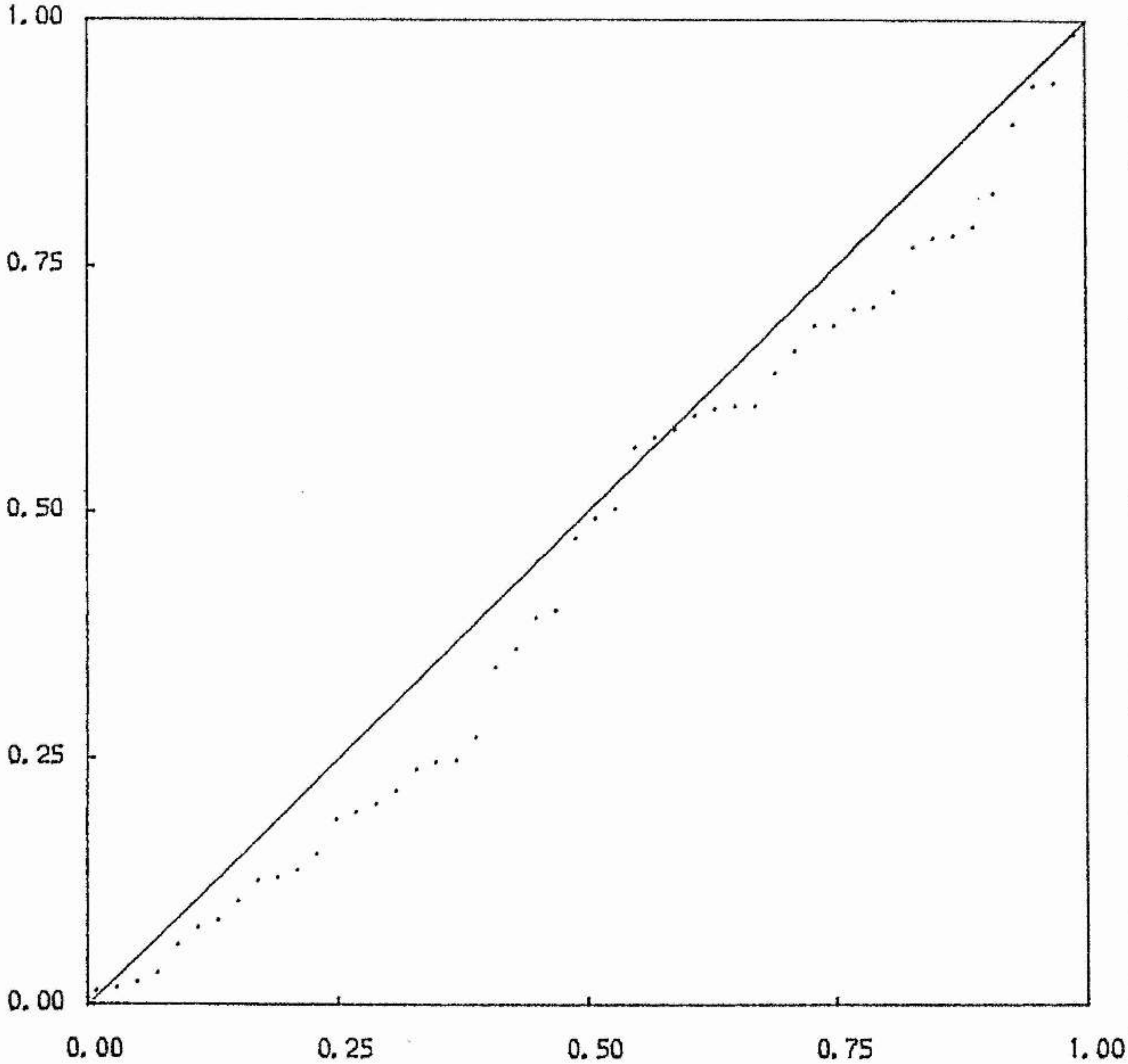


Figure (17)
The longitude plot for sample 1 of Fisher distribution (ii).

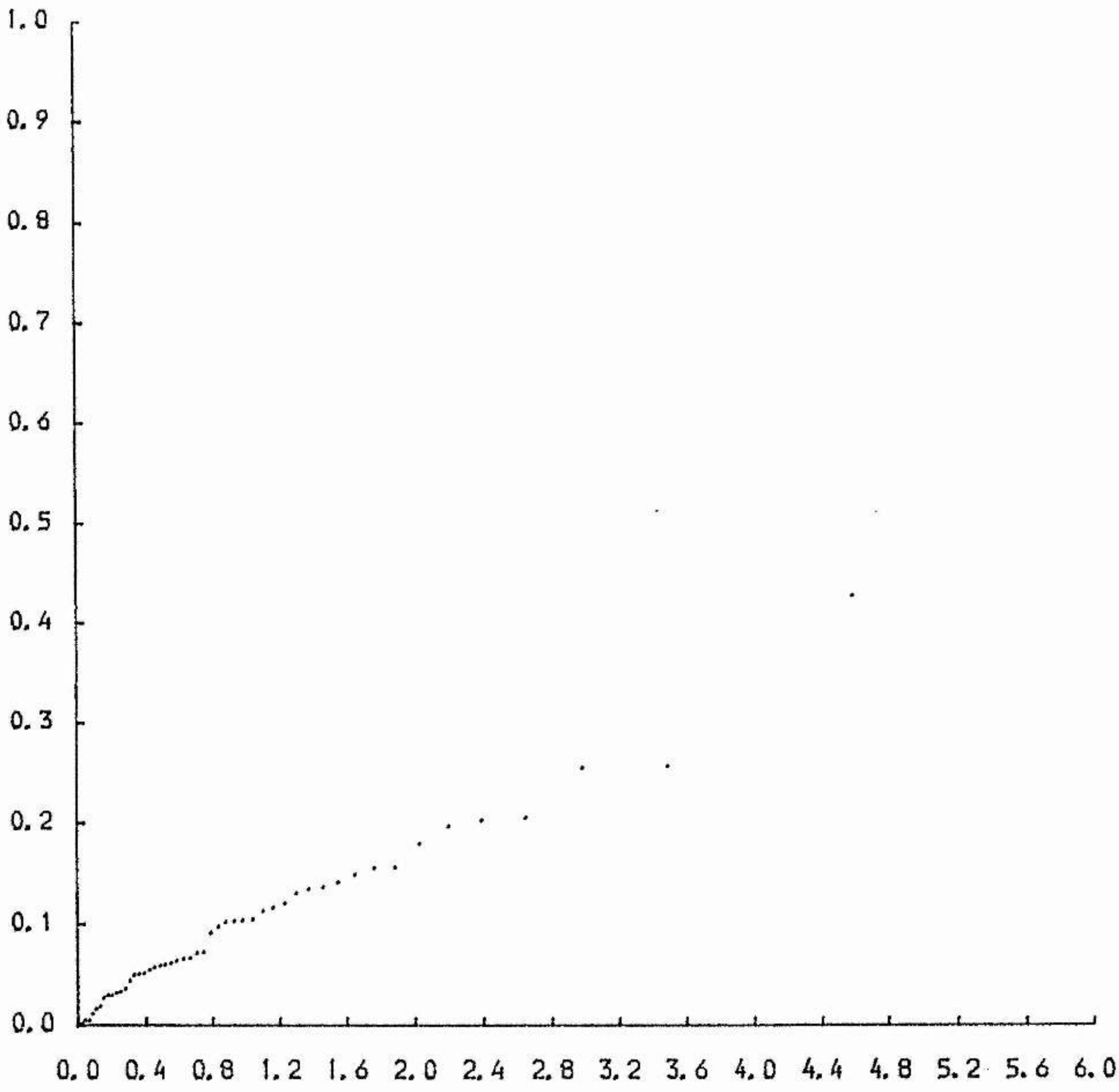


Figure (18)
The colatitude plot for sample 2 of Fisher distribution (ii).

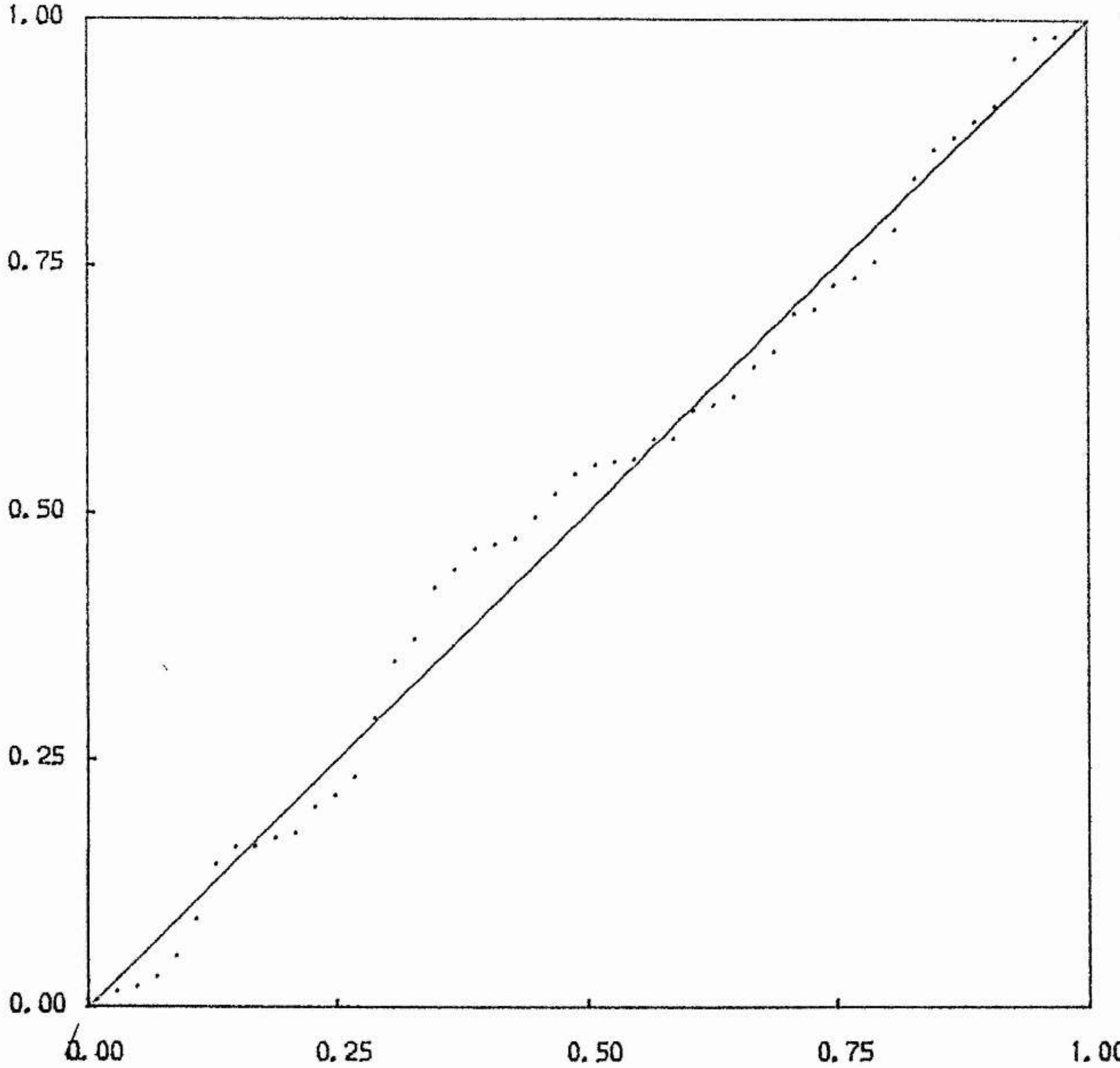


Figure (19)
The longitude plot for sample 2 of Fisher distribution (ii).

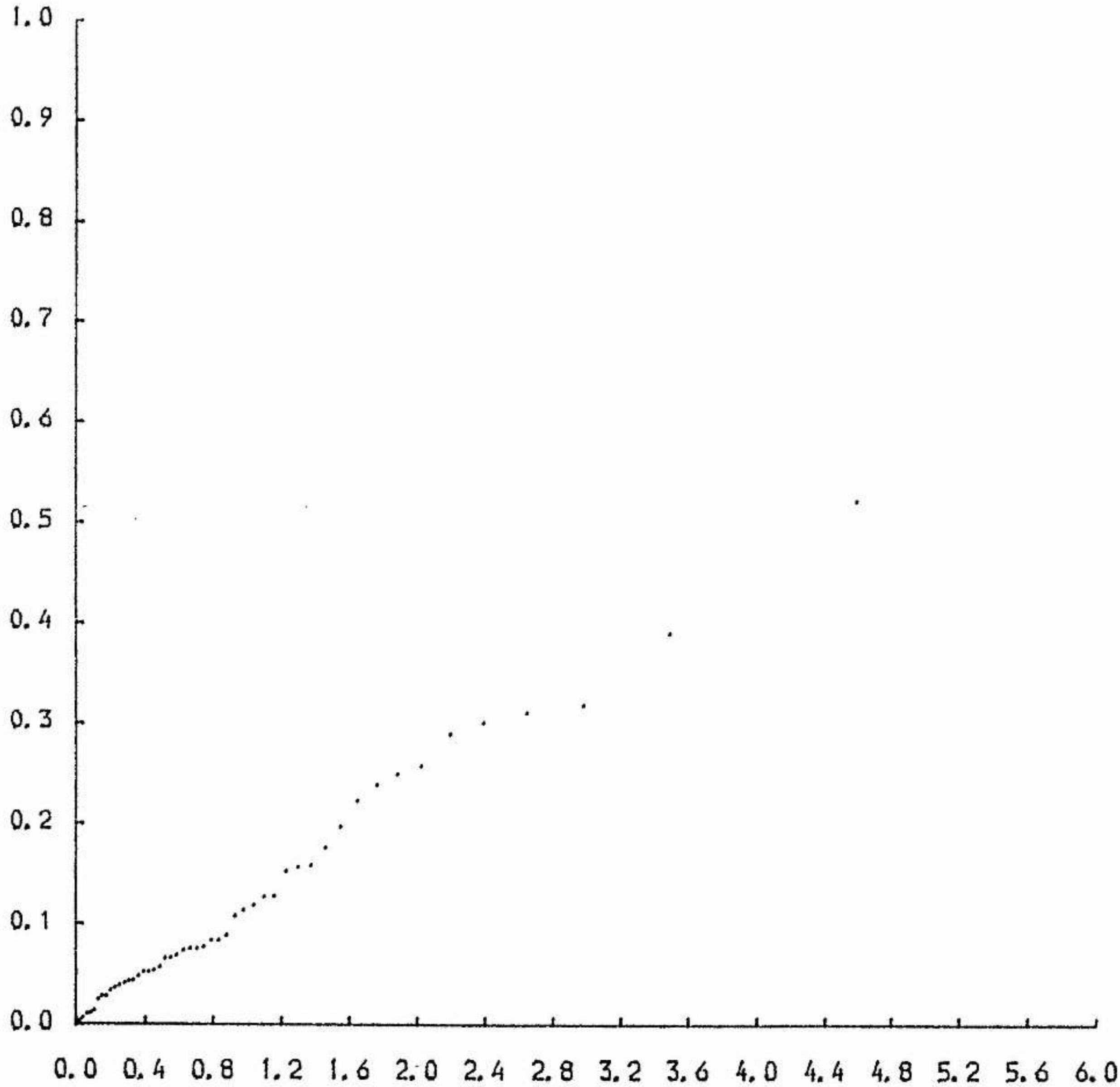


Figure (20)
The colatitude plot for sample 3 of Fisher distribution (ii).

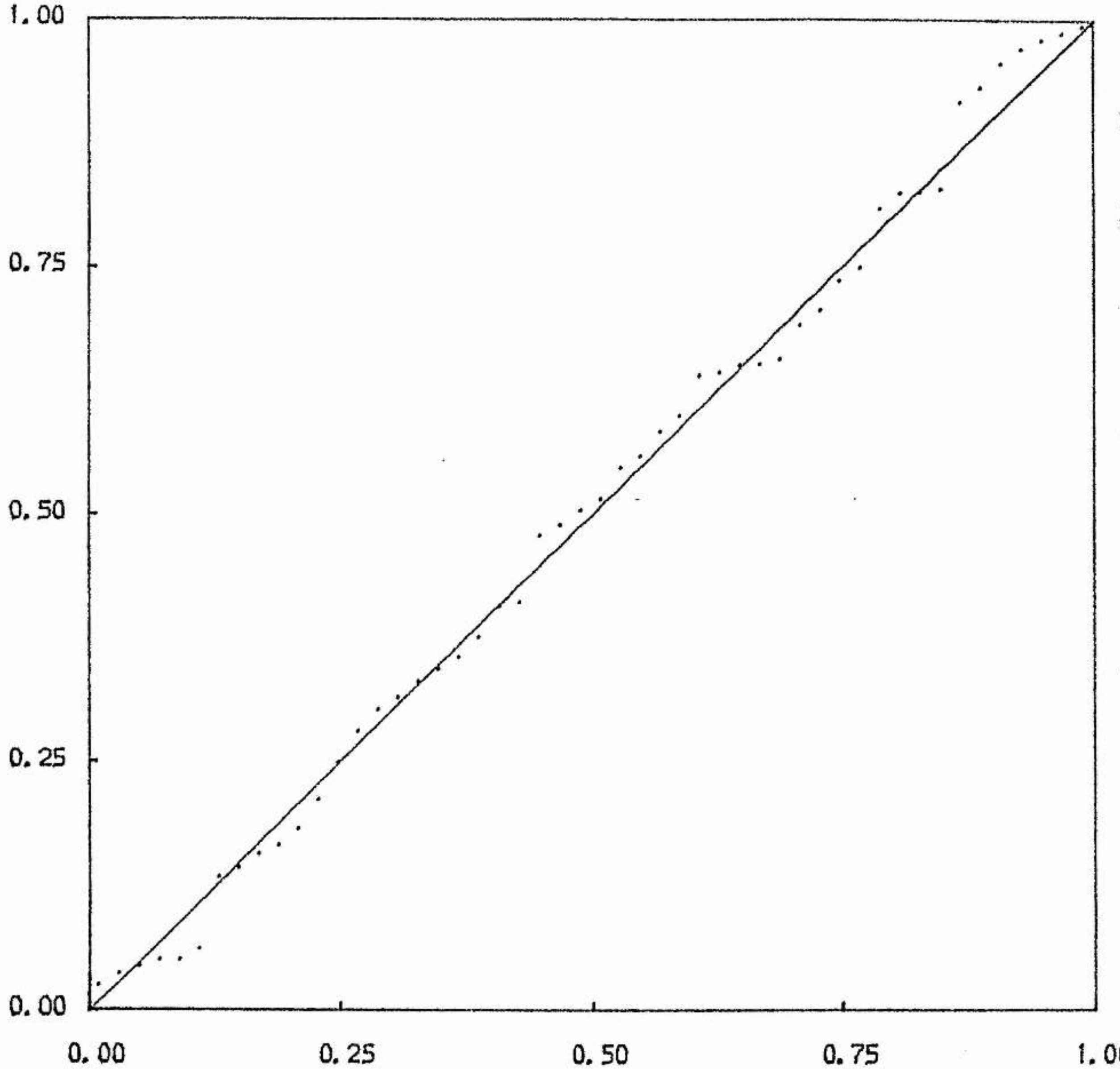


Figure (21)
The longitude plot for sample 3 of Fisher distribution (ii).

5.4 Mixtures of Fisher Distributions

The probability density function of a mixture of two Fisher distributions is given by

$$h(\theta,\phi) = p\, h_1(\theta,\phi) + (1 - p)\, h_2(\theta,\phi)\ , \qquad 0 < p < 1\ ,$$

$$0 \le \theta \le \pi\, , 0 \le \phi < 2\pi\, . \qquad (5.14)$$

where

$$h_i(\theta,\phi) = [\kappa_i/(4\,\pi\, \sinh\kappa_i)]\exp[\kappa_i\{\sin\theta\, \sin\alpha_i \cos(\phi\text{-}\beta_i\,) + \cos\theta \cos\, \alpha_i\}]\, \sin\theta\ ,$$

$$i = 1,2\ , 0 \le \alpha_i \le \pi\, , 0 \le \beta_i\ \le\ 2\,\pi\, , \kappa_i > 0, \quad (5.15)$$

is the Fisher pdf discussed in Section (5.1). p is again the mixing parameter.

The mixture of two Fisher distributions has received little attention in the literature. Stephens(1969) used a method based on maximum likelihood to estimate the unknown parameters in a mixture of two Fisher distributions with opposite modal vectors where the pdf is given by

$$f(\theta) = (\kappa \sin\theta\, /\, 2\, \sinh\kappa)\ \{\ p\ \exp(\kappa \cos\theta) + (1 - p)\, \exp(-\ \kappa\cos\theta)\ \}\ \ ,$$

$$0 \le \theta\ \le \pi \qquad (5.16)$$

and ϕ has a uniform distribution on $(0\ , 2\ \pi)$.

Model (5.16) is used to describe directed data with unequal modes. When $p = 0.5$ in (5.16) the pdf reduces to

$$f(\theta) = (\kappa \sin\theta\, /\, 2\, \sinh\kappa)\ \{\ \cosh(\kappa \cos\theta)\ \}\ \ ,\ 0 \le \theta\ \le \pi \qquad (5.17)$$

This model is used in analysing axial data (see Stephens(1969) for further details).

Wood(1982) also discussed the problems involved in estimating the seven unknown parameters in the mixture of two Fisher distributions. He concludes that the general mixture is awkward to use and the seven parameters have to be estimated numerically. Consequently , Wood proposed a bimodal distribution on the sphere having the same concentration about two directions in roughly equal proportions.

The Wood distribution has pdf

$$g((\theta,\phi);\ (\alpha,\beta),\kappa) = C_F\ \exp[\kappa(\sin\theta\ \sin\alpha\ \cos(2\phi-\beta) + \cos\theta\ \cos\alpha)]\ \sin\theta,$$

$$0 \le \theta \le \pi\ ,\ 0 \le \phi \le 2\pi\ ,\ \kappa > 0. \qquad (5.18)$$

where C_F is given by (5.2).

The only difference between (5.18) and the ordinary Fisher distribution (5.1) is that $\cos(\phi-\beta)$ is now replaced by $\cos(2\phi-\beta)$.

When $\kappa = 0$,(5.18) reduces to the uniform distribution on the sphere. For further details and properties of the distribution see Wood(1982).

Wood also applied model (5.18) to a data set given by Schmidt(1976). For this data a Fisher model (5.1) was found not to be appropriate. In chapter 7 we apply our methods to the Schmidt data and compare our results with Wood's.

5.5 Simulation of Mixtures of Fisher Distributions

To simulate random samples of size n from a mixture of Fisher distributions we do the following :

(1) simulate n observations from each of $f((\theta,\phi);(\alpha_1,\beta_1),\kappa_1)$ and $f((\theta,\phi);(\alpha_2,\beta_2),\kappa_2)$ using the method described in Section (5.2).

(2) choose the required value of the mixing parameter, p.

(3) generate n random numbers from U(0,1) , say, u1, u2, ..., un.

(4) finally we create a sample of n observations from the specified mixture distribution by applying the same procedure as given in Section (2.6) i.e.

if u1 < p then the first observation from the simulated $f((\theta,\phi);(\alpha_1,\beta_1),\kappa_1)$ will be taken into the mixture and if u1 > p then the first observation from the simulated $f((\theta,\phi);(\alpha_2,\beta_2),\kappa_2)$ will be taken into the mixture.

This process is repeated using the remaining random numbers u2, u3,.........., un to obtain the required sample from the mixture distribution.

CHAPTER 6

METHODS OF ESTIMATION

6.1 Maximum Likelihood Estimation

For a single Fisher distribution the maximum likelihood estimates of the mean direction (α,β) and κ are well documented. The maximum likelihood estimate of (α,β) is just the sample mean direction $(\hat{\alpha}_o, \hat{\beta}_o)$ given by (5.5) with (5.6) and (5.7).

The maximum likelihood estimate of κ is then the solution of the equation

$$\coth(\kappa) - \frac{1}{\kappa} = \frac{R}{N} \tag{6.1}$$

Tabulated values of the solution of equation (6.1) are given in Appendix A10 of Fisher, Lewis and Embleton (1987). Fisher(1982) and Fisher, Lewis and Embleton (1987) note that the solution of (6.1) is well approximated by

$$\hat{\kappa} = n \,/\, (n - R) \equiv n \,/\, \sum_{i=1}^{n} C_i \tag{6.2}$$

for $(R/n) \geq 0.95$, a common occurrence in practice.

Here $C_i = 1 - \cos\theta_i'$ and θ_i' is measured relative to the sample mean direction $(\hat{\alpha}_o, \hat{\beta}_o)$ by using the rotation (5.8).

For the mixture of two Fisher distributions (5.14) the normal equations can be written down but not solved explicitly.

As noted in Section(5.4), Stephens(1969) and Wood(1982) used maximum likelihood for the simpler models they considered. In our case we found it simpler to maximise the log-likelihood function using a standard optimisation routine from the NAG Library. The results obtained are discussed in Chapter 7.

6.2 Minimum Distance Estimation

Here we extend the minimum distance method introduced in Section (3.3) to the mixture of two Fisher distributions. Again the measure will be based on the difference between the theoretical and empirical characteristic functions i.e.

$$\left|\Phi_n(\underset{\sim}{t}) - \Phi(\underset{\sim}{t})\right|^2. \tag{6.3}$$

Here $\Phi_n(\underset{\sim}{t})$ is the empirical characteristic function and is defined as follows :

$$\Phi_n(\underset{\sim}{t}) = \frac{1}{n} \sum_{j=1}^{n} \exp(i\, \underset{\sim}{t}'\, \underset{\sim}{l_j}) \tag{6.4}$$

where $\underset{\sim}{t}' = (t_1, t_2, t_3)$

and $\underset{\sim}{l_j}' = (l_j, m_j, n_j) = (\sin\theta_j \cos\phi_j , \sin\theta_j \sin\phi_j , \cos\theta_j)$ are the direction cosines.

$\Phi(\underset{\sim}{t})$ is the characteristic function of the model and is given by

$$\Phi(\underset{\sim}{t}) = p\,\frac{c(\kappa_1)}{c(w_1)} + (1-p)\,\frac{c(\kappa_2)}{c(w_2)} \quad , \text{[see Mardia(1975b)]} \tag{6.5}$$

where $c(\kappa_\lambda) = [\, \kappa_\lambda^{1/2} / \{ (2\pi)^{3/2}\, I_{1/2}(\kappa_\lambda) \} \,]$,

$c(w_\lambda) = [\, w_\lambda^{1/2} / \{ (2\pi)^{3/2}\, I_{1/2}(w_\lambda) \} \,]$

with $w_\lambda = (\kappa_\lambda^2 - \underset{\sim}{t}'\underset{\sim}{t} + 2\, i\, \kappa_\lambda\, \underset{\sim}{t}'\underset{\sim}{\mu_\lambda})^{1/2}$

and $\underset{\sim}{\mu}_\lambda' = (\sin\alpha_\lambda \cos\beta_\lambda , \sin\alpha_\lambda \sin\beta_\lambda , \cos\alpha_\lambda)$ for $\lambda = 1, 2$.

$I_q(\ .\ )$ is again the modified Bessel function of the first kind and order q.

Since $I_{1/2}(\eta) = (\frac{2}{\pi\eta})^{1/2} \sinh\eta$ (see Abramowitz & Stegun p. 443)

we can simplify $\dfrac{c(\kappa_\lambda)}{c(w_\lambda)}$, $\lambda = 1, 2$, as follows.

We have $c(\kappa_\lambda) = [\, \kappa_\lambda^{1/2} / \{ (2\pi)^{3/2} I_{1/2}(\kappa_\lambda) \} \,] = \dfrac{\kappa_\lambda}{4\pi \sinh \kappa_\lambda}$ and

$$c(w_\lambda) = [w_\lambda^{1/2} / \{ (2\pi)^{3/2} I_{1/2}(w_\lambda) \} \,] = \frac{w_\lambda}{4\pi \sinh w_\lambda} .$$

Now

$$w_\lambda^2 = (\kappa_\lambda^2 - \underset{\sim}{t}' \underset{\sim}{t} + 2\, i\, \kappa_\lambda \underset{\sim}{t}' \underset{\sim}{\mu}_\lambda)$$

if we put $w_\lambda^2 = r_\lambda \exp(i\,\psi_\lambda)$ then we have

$$r_\lambda \cos\psi_\lambda = \kappa_\lambda^2 - \underset{\sim}{t}' \underset{\sim}{t} \quad , \quad r_\lambda \sin\psi_\lambda = 2\kappa_\lambda \underset{\sim}{t}' \underset{\sim}{\mu}_\lambda$$

i.e. $$\tan\psi_\lambda = \frac{2\kappa_\lambda \underset{\sim}{t}' \underset{\sim}{\mu}_\lambda}{(\kappa_\lambda^2 - \underset{\sim}{t}' \underset{\sim}{t})} ,$$

and $r_\lambda^2 = (\kappa_\lambda^2 - \underset{\sim}{t}' \underset{\sim}{t})^2 + (2\kappa_\lambda \underset{\sim}{t}' \underset{\sim}{\mu}_\lambda)^2$

so $w_\lambda = r_\lambda^{1/2} \exp(i(\frac{\psi_\lambda}{2})) = r_\lambda^{1/2} \cos(\frac{\psi_\lambda}{2}) + i\, r_\lambda^{1/2} \sin(\frac{\psi_\lambda}{2})$

then $\sinh w_\lambda = \sinh(r_\lambda^{1/2} \cos (\frac{\Psi_\lambda}{2}) + i\ r_\lambda^{1/2} \sin (\frac{\Psi_\lambda}{2}))$

$$= \sinh(r_\lambda^{1/2} \cos (\frac{\Psi_\lambda}{2})) \cosh(i\ r_\lambda^{1/2} \sin (\frac{\Psi_\lambda}{2}))$$

$$+ \cosh(r_\lambda^{1/2} \cos (\frac{\Psi_\lambda}{2})) \sinh(i\ r_\lambda^{1/2} \sin (\frac{\Psi_\lambda}{2}))$$

$$= \sinh(r_\lambda^{1/2} \cos (\frac{\Psi_\lambda}{2})) \cos(r_\lambda^{1/2} \sin (\frac{\Psi_\lambda}{2}))$$

$$+ i \cosh(r_\lambda^{1/2} \cos (\frac{\Psi_\lambda}{2})) \sin(r_\lambda^{1/2} \sin (\frac{\Psi_\lambda}{2}))$$

Now put

$$x_\lambda = r_\lambda^{1/2} \cos (\frac{\Psi_\lambda}{2}) \ , \quad y_\lambda = r_\lambda^{1/2} \sin (\frac{\Psi_\lambda}{2}) \quad , \ \lambda = 1, 2$$

then

$$\frac{c(\kappa_\lambda)}{c(w_\lambda)} = \frac{\kappa_\lambda \sinh w_\lambda}{w_\lambda \sinh \kappa_\lambda} = \frac{\kappa_\lambda[\sinh(x_\lambda) \cos(y_\lambda) + i \cosh(x_\lambda) \sin(y_\lambda)]}{(x_\lambda + i\ y_\lambda) \sinh \kappa_\lambda}$$

$$= \frac{(x_\lambda - i\ y_\lambda)}{(x_\lambda - i\ y_\lambda)} \ \frac{\kappa_\lambda[\sinh(x_\lambda) \cos(y_\lambda) + i \cosh(x_\lambda) \sin(y_\lambda)]}{(x_\lambda + i\ y_\lambda) \sinh \kappa_\lambda}$$

$$= \frac{\kappa_\lambda(x_\lambda - i\ y_\lambda)[\sinh(x_\lambda) \cos(y_\lambda) + i \cosh(x_\lambda) \sin(y_\lambda)]}{r_\lambda \sinh \kappa_\lambda}$$

$$= \frac{\kappa_\lambda}{r_\lambda \sinh \kappa_\lambda} [\{ x_\lambda \sinh(x_\lambda) \cos(y_\lambda) + y_\lambda \cosh(x_\lambda) \sin(y_\lambda) \}$$

$$+ i \{ x_\lambda \cosh(x_\lambda) \sin(y_\lambda) - y_\lambda \sinh(x_\lambda) \cos(y_\lambda) \}] \qquad (6.6)$$

Now

$$\left|\Phi_n(\underset{\sim}{t}) - \Phi(\underset{\sim}{t})\right|^2 = \Big|\Big[\frac{1}{n}\sum_{j=1}^{n} \cos \underset{\sim}{t}' \underset{\sim}{l_j} + i\,\frac{1}{n}\sum_{j=1}^{n} \sin \underset{\sim}{t}' \underset{\sim}{l_j}\Big]$$

$$- \Big[\, p\,\frac{c(\kappa_1)}{c(w_1)} + (1-p)\,\frac{c(\kappa_2)}{c(w_2)}\Big]\Big|^2 \quad \text{where } \frac{c(\kappa_\lambda)}{c(w_\lambda)}$$

is given by (6.6) for $\lambda = 1, 2$.

We can simplify this expression by letting

$$\gamma_1 = x_1 \sinh x_1 \cos y_1 + y_1 \cosh x_1 \sin y_1$$

$$\gamma_2 = x_2 \sinh x_2 \cos y_2 + y_2 \cosh x_2 \sin y_2$$

$$\xi_1 = x_1 \cosh x_1 \sin y_1 - y_1 \sinh x_1 \cos y_1$$

$$\xi_2 = x_2 \cosh x_2 \sin y_2 - y_2 \sinh x_2 \cos y_2$$

Hence

$$\left|\Phi_n(\underset{\sim}{t}) - \Phi(\underset{\sim}{t})\right|^2 = \Big|\Big[\frac{1}{n}\sum_{j=1}^{n} \cos \underset{\sim}{t}' \underset{\sim}{l_j} + i\,\frac{1}{n}\sum_{j=1}^{n} \sin \underset{\sim}{t}' \underset{\sim}{l_j}\Big]$$

$$- \Big[\, \frac{p\,\kappa_1}{r_1 \sinh \kappa_1}(\gamma_1 + i\,\xi_1) + \frac{(1-p)\,\kappa_2}{r_2 \sinh \kappa_2}(\gamma_2 + i\,\xi_2)\Big]\Big|^2$$

$$= \Big[\frac{1}{n}\sum_{j=1}^{n} \cos \underset{\sim}{t}' \underset{\sim}{l_j} - \Big\{\frac{p\,\kappa_1}{r_1 \sinh \kappa_1}(\gamma_1) + \frac{(1-p)\,\kappa_2}{r_2 \sinh \kappa_2}(\gamma_2)\Big\}\Big]^2$$

$$+ \Big[\frac{1}{n}\sum_{j=1}^{n} \sin \underset{\sim}{t}' \underset{\sim}{l_j} - \Big\{\frac{p\,\kappa_1}{r_1 \sinh \kappa_1}(\xi_1) + \frac{(1-p)\,\kappa_2}{r_2 \sinh \kappa_2}(\xi_2)\Big\}\Big]^2 . \quad (6.7)$$

Minimisation of (6.7) is not as simple in three dimensions as it was in two dimensions. In two dimensions t only took integer values and we were able to simplify (3.8) using the von Neumann addition formula. Here $\underset{\sim}{t}$ is a vector quantity and so summation or integration of (6.7) over a suitable range of values of $\underset{\sim}{t}$ is not going to be an easy task. An alternative way is to minimise (6.7) for selected values of $\underset{\sim}{t}$. Selection of these values is arbitrary but it seems sensible to make them as simple as possible. Consequently we chose the seven values (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1) and (1,1,1). We need seven values for $\underset{\sim}{t}$ since there are seven parameters to be estimated. Using these values for $\underset{\sim}{t}$ and a standard NAG optimisation routine we minimised (6.7) and achieved convergence in most cases.

CHAPTER 7

DISCUSSION OF RESULTS

In Section (7.1) we demonstrate our results on the simulated data and in section (7.2) on the real data.

Again the minimization subroutines require starting values for the seven parameters κ_1 , κ_2 , α_1 , α_2 , β_1 , β_2 and p.

Wood(1982) gave a method for estimating the parameters α_1 , α_2 , β_1 , β_2 and κ for the bimodal distribution (5.18) . Our starting values procedure uses this method to obtain estimates of α_1 , α_2 , β_1 , β_2 and κ_1. We then set $\kappa_1 = \kappa_2$ and $p = 0.5$ to obtain starting values for κ_2 and p.

The procedure is as follows :

(1) To obtain the estimates of α_1 , α_2 , β_1 and β_2 we

(i) Define the vectors

$$\underset{\sim}{\mu}_1' = (\cos\gamma\cos\delta, \cos\gamma\sin\delta, -\sin\gamma)$$

$$\underset{\sim}{\mu}_2' = (-\sin\delta , \cos\delta, 0) \tag{7.1}$$

$$\underset{\sim}{\mu}_3' = (\sin\gamma\cos\delta, \sin\gamma\sin\delta, \cos\gamma)$$

(ii) Define

$$U = \sum_{i=1}^{n} \underset{\sim}{X}_i' \, \underset{\sim}{\mu}_3$$

$$V = \sum_{i=1}^{n} \{ (\underset{\sim}{X}_i' \, \underset{\sim}{\mu}_1)^2 - (\underset{\sim}{X}_i' \, \underset{\sim}{\mu}_2)^2 \} \; / \; \{ 1 - (\underset{\sim}{X}_i' \, \underset{\sim}{\mu}_3)^2 \}^{1/2} \qquad (7.2)$$

$$W = \sum_{i=1}^{n} 2 (\underset{\sim}{X}_i' \, \underset{\sim}{\mu}_1)(\underset{\sim}{X}_i' \, \underset{\sim}{\mu}_2) \; / \; \{ 1 - (\underset{\sim}{X}_i' \, \underset{\sim}{\mu}_3)^2 \}^{1/2},$$

where $\underset{\sim}{X}_i' = (x_i , y_i , z_i)$,

with $x_i = \sin \theta_i \cos \phi_i$, $y_i = \sin \theta_i \sin \phi_i$, $z_i = \cos \theta_i$.

(iii) Define

$$S^2(\gamma,\delta) = U^2 + V^2 + W^2 \qquad (7.3)$$

(iv) Maximize $S^2(\gamma,\delta)$ to get an estimate of γ and δ , say $\hat{\gamma}$ and $\hat{\delta}$. To maximize $S^2(\gamma,\delta)$ we used a subroutine [E04JAF] which is available in the NAG Library.

This routine requires initial values for γ and δ . To obtain initial values of γ and δ we use the sample mean direction(γ_R, δ_R) which is calculated by using (5.5), (5.6) and (5.7) with $\hat{\alpha}_\circ = \gamma_R$ and $\hat{\beta}_\circ = \delta_R$.

(v) Substitute $\hat{\gamma}$ and $\hat{\delta}$ into the set of equations (7.2) to obtain U , V and W . Then estimate α and β as follows :

(a) Set $U'' = \dfrac{U}{S(\hat{\gamma},\hat{\delta})}$

(b) Set $V'' = \dfrac{V}{S(\hat{\gamma},\hat{\delta})}$

(c) Set $W'' = \dfrac{W}{S(\hat{\gamma},\hat{\delta})}$

(d) $\alpha = \cos^{-1}(U'')$, $\beta = \tan^{-1}(\dfrac{W''}{V''})$.

(vi) Now to find $\hat{\alpha}_1$, $\hat{\beta}_1$, $\hat{\alpha}_2$, $\hat{\beta}_2$ we do the following

(a) Set $(\alpha_1, \beta_1) = (\alpha, \frac{1}{2}\beta)$ and $(\alpha_2, \beta_2) = (\alpha, \frac{1}{2}\beta + \pi)$.

(b) Calculate $\hat{\underset{\sim}{\mu}}_1$, $\hat{\underset{\sim}{\mu}}_2$, $\hat{\underset{\sim}{\mu}}_3$ from (7.1) using $\hat{\gamma}$, $\hat{\delta}$ and then form the 3×3 matrix $A = (\hat{\underset{\sim}{\mu}}_1 , \hat{\underset{\sim}{\mu}}_2 , \hat{\underset{\sim}{\mu}}_3)$

(c) Transform the estimates (α_1, β_1) , (α_2, β_2) to $(\hat{\alpha}_1 , \hat{\beta}_1)$, $(\hat{\alpha}_2 , \hat{\beta}_2)$ respectively by using

$$\begin{pmatrix} \sin\hat{\alpha}_i \cos\hat{\beta}_i \\ \sin\hat{\alpha}_i \sin\hat{\beta}_i \\ \cos\hat{\alpha}_i \end{pmatrix} = A \begin{pmatrix} \sin\alpha_i \cos\beta_i \\ \sin\alpha_i \sin\beta_i \\ \cos\alpha_i \end{pmatrix}, \quad i = 1,2$$

$(\hat{\alpha}_1 , \hat{\beta}_1)$ and $(\hat{\alpha}_2 , \hat{\beta}_2)$ are then the required estimates.

(2) To obtain the estimate of κ_1 we use (6.2) with

$(\hat{\alpha}_0 , \hat{\beta}_0) = (\gamma_R, \delta_R)$ in (5.8).

7.1 The Simulated data

As in Section (4.1) we compare results for single samples and sets of samples for the different methods of estimation discussed in chapter 6. We again represent maximum likelihood estimation by MLE and minimum distance estimation based on the characteristic function by MDE.

Tables 44 - 47 display the single sample results for the four simulated data sets which we shall use throughout this section. These are :

Example 7 : $p = 0.25, \kappa_1 = 3.5, \kappa_2 = 6.5, \alpha_1 = 50, \beta_1 = 120,$
$\alpha_2 = 100, \beta_2 = 320.$

Example 8 : $p = 0.60, \kappa_1 = 8.0, \kappa_2 = 1.5, \alpha_1 = 35, \beta_1 = 75,$
$\alpha_2 = 130, \beta_2 = 220.$

Example 9 : $p = 0.45, \kappa_1 = 9.5, \kappa_2 = 6.0, \alpha_1 = 100, \beta_1 = 120,$
$\alpha_2 = 55, \beta_2 = 300.$

Example 10 : $p = 0.9, \kappa_1 = 4.0, \kappa_2 = 2.5, \alpha_1 = 70, \beta_1 = 140,$
$\alpha_2 = 145, \beta_2 = 260.$

These data sets were generated using the procedure described in Section(5.5). Both methods of estimation introduced in chapter 6 were implemented for the above mixtures.

We again make use of condition (4.6) except now $\hat{p}(\text{Ini})$ is the estimate of p when the initial values are obtained from the procedure described earlier in this chapter.

From tables 44 - 47 we see that increasing the sample size n usually causes a slight improvement in the results more so for MLE than for MDE. Also, it is clear that the estimates of the κ's are very variable for small samples but get closer to the true values as n increases. Apart from table 47 the estimates for the modes (α_1, β_1) and (α_2, β_2) are close to the true values. The results obtained in table 47 for example 10 are much more variable than for the other examples and this is to be expected since the value of p is close to 1.

The CPU time varies from method to method but usually MDE is quicker than MLE except for small sample sizes ($n = 50$) and $n = 100$ in table 44.

In fact, according to tables 44 and 46 there is little to choose between MLE and MDE although MLE is slightly better than MDE for the other tables. Consequently, one method does not seem to be better than the other overall. However, since we are only considering single samples here we must be cautious about drawing conclusions in this situation.

To compare the results for sets of samples we again simulated 50 and 200 samples of sizes 50, 100 and 200 for each of the four examples given earlier.Each of the six sets of results is described by a single table which gives the results for MDE and MLE.

The bias and MSE for each parameter were calculated for each set of samples. Samples which did not satisfy condition (4.6) were noted but excluded from the bias and MSE calculations. Samples where one of the κ's reached the upper limit were also excluded.

Each table gives the bias and MSE and also includes the number of samples satisfying condition (4.6) , C1, the number of samples which converge but do not satisfy condition (4.6), C2, and the number of samples where one of the κ's reaches the upper limit, C3.

It is clear from tables 48 - 53 of example 7, tables 54 - 59 of example 8 , tables 60 - 65 of example 9 and tables 66 - 71 of example 10 that , in most cases, MLE performs better than MDE as far as bias and MSE are concerned. Also, except for three cases (see tables 56, 57, 59) , the number of samples which satisfy condition (4.6) when using MLE is greater than that when using MDE. Again, in most cases, the CPU time taken when using MDE will be less than that when using MLE.

When the sample size n is increased the bias is reduced in most cases for all the examples while the MSE is reduced in all cases for all the examples except example 10. For example 10 there are a few cases where the MSE is not reduced. Also, for all cases the number of samples satisfying condition (4.6) is increased whilst the number where one of the κ's reaches the limit is decreased.

Increasing the number of samples S usually leads to an increase in the bias and MSE for some parameters and a decrease for others.

As in the circular case (see Section 3) we sometimes have an overflow problem with the exponential function (used here in the definition of the sinh function). Again this tends to happen with small samples when the mixing parameter p is near one or zero. This problem has more of an effect on MDE than on MLE. In fact, as n increases MLE converges almost all the time.

Also, the iterative procedure sometimes stops before convergence to the final estimates is achieved. The procedure can be restarted by using the last estimates given before stopping as the initial estimates in a new optimisation. This can be repeated until final convergence is attained.

Table 44

n=50

	MLE	MDE
P	0.4039	0.2961
κ_1	1.2426	2.6883
κ_2	12.8789	9.2363
α_1	19.8488	43.7699
β_1	130.2033	135.8277
α_2	100.0223	98.7401
β_2	323.8345	323.5916
CPU	21.31	36.35

n=100

	MLE	MDE
P	0.2846	0.2511
κ_1	2.9613	4.4179
κ_2	7.6673	7.3947
α_1	34.0773	39.5840
β_1	124.8504	129.9775
α_2	98.0336	97.2202
β_2	325.2814	325.8253
CPU	30.97	37.43

n=200

	MLE	MDE
P	0.2666	0.2585
κ_1	3.2023	3.8258
κ_2	6.5667	7.1143
α_1	41.7912	42.8466
β_1	120.3055	124.9941
α_2	97.0375	97.1754
β_2	322.4529	323.0871
CPU	62.6	38.82

The true values are : $p = 0.25$, $\kappa_1 = 3.5$, $\kappa_2 = 6.5$, $\alpha_1 = 50$, $\beta_1 = 120$, $\alpha_2 = 100$, $\beta_2 = 320$.

Table 45

n=50

	MLE	MDE
P	0.5820	0.6237
κ_1	6.9635	5.5946
κ_2	2.7668	3.7464
α_1	30.6123	32.1526
β_1	90.8009	91.5254
α_2	133.5276	134.5845
β_2	210.6618	217.1955
CPU	19.35	21.19

n=100

	MLE	MDE
P	0.6049	0.6216
κ_1	8.2768	7.3545
κ_2	2.1676	2.4037
α_1	29.5461	30.7100
β_1	90.2670	90.8821
α_2	129.6611	129.4461
β_2	221.5569	225.7264
CPU	30.42	20.29

n=200

	MLE	MDE
P	0.6854	0.6814
κ_1	7.38	7.5849
κ_2	2.1406	2.0669
α_1	31.6044	32.1767
β_1	82.8060	83.6678
α_2	125.2240	124.9007
β_2	221.1729	223.6276
CPU	58.40	22.02

The true values are : $p = 0.6$, $\kappa_1 = 8.0$, $\kappa_2 = 1.5$, $\alpha_1 = 35$, $\beta_1 = 75$, $\alpha_2 = 130$, $\beta_2 = 220$.

Table 46

n=50

	MLE	MDE
P	0.5001	0.4977
κ_1	7.9372	8.1940
κ_2	7.6656	7.5734
α_1	97.8444	97.4582
β_1	125.4314	127.0705
α_2	52.9765	53.9688
β_2	297.4772	299.7178
CPU	13.54	7.89

n=100

	MLE	MDE
P	0.4700	0.4788
κ_1	9.1079	8.2346
κ_2	7.0702	7.7952
α_1	95.7043	94.2630
β_1	123.7563	124.4098
α_2	52.2755	53.9013
β_2	305.2041	306.1784
CPU	31.84	8.98

n=200

	MLE	MDE
P	0.4599	0.4609
κ_1	8.7033	9.0095
κ_2	6.4586	6.7325
α_1	96.4135	95.5712
β_1	121.1283	121.5875
α_2	52.4011	53.4828
β_2	303.8009	304.3355
CPU	60.07	8.64

The true values are : $p = 0.45$, $\kappa_1 = 9.5$, $\kappa_2 = 6.0$, $\alpha_1 = 100$, $\beta_1 = 120$, $\alpha_2 = 55$, $\beta_2 = 300$.

Table 47

n=50

	MLE	MDE
P	0.8888	0.7592
κ_1	4.4549	4.4114
κ_2	7.1105	2.4378
α_1	63.2419	65.0601
β_1	143.1411	141.0716
α_2	157.9820	146.7968
β_2	283.9870	147.8360
CPU	28.51	29.01

n=100

	MLE	MDE
P	0.9231	0.7600
κ_1	4.5462	4.5430
κ_2	5.6972	1.7675
α_1	62.3903	64.8949
β_1	146.7654	143.0693
α_2	159.7142	118.7247
β_2	301.6073	186.9896
CPU	60.72	34.68

n=200

	MLE	MDE
P	0.9136	0.7123
κ_1	4.2162	5.7559
κ_2	3.6899	1.3457
α_1	63.0135	66.3110
β_1	143.7674	145.0869
α_2	161.5152	116.5242
β_2	256.3448	150.2818
CPU	96.59	32.41

The true values are : $p = 0.9$, $\kappa_1 = 4.0$, $\kappa_2 = 2.5$, $\alpha_1 = 70$, $\beta_1 = 140$, $\alpha_2 = 145$, $\beta_2 = 260$.

EXAMPLE 7

Table 48 : n=50 , S=50

		MLE	MDE
p	BIAS	0.0031	- 0.0029
p	MSE	0.0037	0.0037
κ_1	BIAS	0.8452	1.7693
κ_1	MSE	5.0402	14.8483
κ_2	BIAS	0.8203	0.6872
κ_2	MSE	2.9904	5.4115
α_1	BIAS	- 1.5363	- 0.2560
α_1	MSE	104.7872	100.4855
β_1	BIAS	- 1.3006	- 0.5302
β_1	MSE	281.4077	479.2135
α_2	BIAS	1.1401	0.5858
α_2	MSE	15.5957	18.9156
β_2	BIAS	- 0.5042	- 0.2909
β_2	MSE	16.4567	16.9804
C_1		48	47
C_2		2	3
C_3		0	0

Table 49 : n=50 , S=200

		MLE	MDE
p	BIAS	0.0033	0.0050
p	MSE	0.0045	0.0049
κ_1	BIAS	0.7361	1.4658
κ_1	MSE	4.1661	12.6168
κ_2	BIAS	0.7412	0.9488
κ_2	MSE	2.8121	7.0912
α_1	BIAS	- 1.5576	- 0.2816
α_1	MSE	145.2807	144.7900
β_1	BIAS	- 1.5794	- 3.2422
β_1	MSE	271.3439	372.9735
α_2	BIAS	0.1889	0.0404
α_2	MSE	16.8903	18.5840
β_2	BIAS	- 0.0564	- 0.3112
β_2	MSE	17.1623	18.2515
C_1		194	178
C_2		6	15
C_3		0	7

EXAMPLE 7

Table 50 : n=100 , S=50

		MLE	MDE
p	BIAS	0.0001	0.0001
p	MSE	0.0015	0.0018
κ_1	BIAS	0.3394	0.7612
κ_1	MSE	0.9430	5.3750
κ_2	BIAS	0.2938	0.3779
κ_2	MSE	0.7974	3.3207
α_1	BIAS	0.9166	0.0833
α_1	MSE	58.2098	70.2144
β_1	BIAS	- 1.1731	- 2.0233
β_1	MSE	100.6729	143.7941
α_2	BIAS	- 0.1189	- 0.1838
α_2	MSE	9.2740	10.2551
β_2	BIAS	- 0.5635	- 0.4627
β_2	MSE	11.0251	11.3805
C_1		50	49
C_2		0	1
C_3		0	0

Table 51 : n=100 , S=200

		MLE	MDE
p	BIAS	0.0016	0.0001
p	MSE	0.0021	0.0026
κ_1	BIAS	0.3561	0.8975
κ_1	MSE	1.3983	6.5055
κ_2	BIAS	0.2712	0.3802
κ_2	MSE	0.9627	3.7158
α_1	BIAS	- 0.2483	- 0.4786
α_1	MSE	59.5655	74.0742
β_1	BIAS	- 0.8718	- 1.7569
β_1	MSE	108.8782	143.9206
α_2	BIAS	- 0.2239	- 0.2550
α_2	MSE	9.0647	9.9164
β_2	BIAS	- 0.1727	- 0.3123
β_2	MSE	8.6254	8.8964
C_1		200	191
C_2		0	9
C_3		0	0

EXAMPLE 7

Table 52 : n=200 , S=50

		MLE	MDE
p	BIAS	- 0.0043	- 0.0053
p	MSE	0.0007	0.0007
κ_1	BIAS	0.3030	0.4109
κ_1	MSE	0.5756	1.3375
κ_2	BIAS	0.0867	0.1058
κ_2	MSE	0.3413	0.8153
α_1	BIAS	0.4388	0.0485
α_1	MSE	29.5535	44.1372
β_1	BIAS	- 0.8664	- 1.0210
β_1	MSE	57.8200	51.2401
α_2	BIAS	- 0.0880	- 0.1149
α_2	MSE	5.7449	5.5019
β_2	BIAS	- 0.2413	- 0.0344
β_2	MSE	4.2592	4.1330
C_1		50	49
C_2		0	1
C_3		0	0

Table 53 : n=200 , S=200

		MLE	MDE
p	BIAS	- 0.0013	- 0.0019
p	MSE	0.0010	0.0013
κ_1	BIAS	0.1612	0.3977
κ_1	MSE	0.5476	1.5549
κ_2	BIAS	0.1816	0.2477
κ_2	MSE	0.4492	1.1612
α_1	BIAS	0.1944	0.0161
α_1	MSE	32.2359	40.6679
β_1	BIAS	- 0.9582	- 1.0739
β_1	MSE	57.2525	70.0913
α_2	BIAS	- 0.0473	- 0.0327
α_2	MSE	4.8071	5.2911
β_2	BIAS	0.0619	0.1619
β_2	MSE	3.7495	4.0619
C_1		200	196
C_2		0	4
C_3		0	0

EXAMPLE 8

Table 54 : n=50 , S=50

		MLE	MDE
p	BIAS	- 0.0056	0.0081
p	MSE	0.0075	0.0082
κ_1	BIAS	0.9900	0.7592
κ_1	MSE	7.2280	17.5937
κ_2	BIAS	0.4515	0.7114
κ_2	MSE	0.8146	1.3594
α_1	BIAS	0.0879	0.2638
α_1	MSE	13.4074	15.3336
β_1	BIAS	0.7845	0.6057
β_1	MSE	53.7238	46.9739
α_2	BIAS	- 0.0660	- 0.1926
α_2	MSE	327.5737	273.1885
β_2	BIAS	- 2.5452	- 1.0153
β_2	MSE	548.9316	515.9531
C_1		49	46
C_2		1	0
C_3		0	4

Table 55 : n=50 , S=200

		MLE	MDE
p	BIAS	- 0.0001	0.0047
p	MSE	0.0066	0.0072
κ_1	BIAS	0.8151	0.9821
κ_1	MSE	7.3176	16.3677
κ_2	BIAS	0.3898	0.5630
κ_2	MSE	0.8484	1.2551
α_1	BIAS	- 0.5580	- 0.2278
α_1	MSE	17.9125	20.8119
β_1	BIAS	0.4367	0.4142
β_1	MSE	51.0360	56.0784
α_2	BIAS	- 0.5635	- 4.4847
α_2	MSE	303.9941	448.6959
β_2	BIAS	- 2.0257	- 5.4980
β_2	MSE	602.3511	935.5557
C_1		194	187
C_2		6	3
C_3		0	10

EXAMPLE 8

Table 56 : n=100 , S=50

		MLE	MDE
p	BIAS	- 0.0057	- 0.0093
p	MSE	0.0034	0.0050
κ_1	BIAS	0.5078	1.5102
κ_1	MSE	2.2598	21.3071
κ_2	BIAS	0.2300	0.2576
κ_2	MSE	0.3001	0.4415
α_1	BIAS	0.0403	- 0.1905
α_1	MSE	12.3857	14.0440
β_1	BIAS	0.3720	0.1643
β_1	MSE	33.9604	29.9087
α_2	BIAS	- 0.2425	- 0.8220
α_2	MSE	152.3356	164.8847
β_2	BIAS	- 2.2313	- 4.0318
β_2	MSE	241.8934	380.0549
C_1		49	50
C_2		1	0
C_3		0	0

Table 57 : n=100 , S=200

		MLE	MDE
p	BIAS	- 0.0025	- 0.0002
p	MSE	0.0033	0.0043
κ_1	BIAS	0.4246	0.7282
κ_1	MSE	2.2670	9.6714
κ_2	BIAS	0.1544	0.2280
κ_2	MSE	0.2847	0.4740
α_1	BIAS	- 0.2584	- 0.1469
α_1	MSE	9.8927	11.1264
β_1	BIAS	- 0.0869	- 0.0262
β_1	MSE	28.6340	32.5443
α_2	BIAS	- 1.7969	- 2.5221
α_2	MSE	167.3964	183.7247
β_2	BIAS	- 1.2630	- 1.0137
β_2	MSE	239.1525	272.9060
C_1		196	197
C_2		4	1
C_3		0	2

EXAMPLE 8

Table 58 : n=200 , S=50

		MLE	MDE
p	BIAS	0.0024	0.0026
p	MSE	0.0016	0.0024
κ_1	BIAS	0.1358	0.4312
κ_1	MSE	0.9397	5.3402
κ_2	BIAS	0.1406	0.1723
κ_2	MSE	0.1395	0.2244
α_1	BIAS	- 0.0652	- 0.1637
α_1	MSE	5.8970	7.4180
β_1	BIAS	0.0790	- 0.2264
β_1	MSE	13.4430	11.2914
α_2	BIAS	- 0.0773	- 0.4483
α_2	MSE	79.4266	93.1940
β_2	BIAS	- 2.4335	- 2.8319
β_2	MSE	132.3426	160.9327
C_1		50	50
C_2		0	0
C_3		0	0

Table 59 : n=200 , S=200

		MLE	MDE
p	BIAS	0.0003	- 0.0032
p	MSE	0.0016	0.0024
κ_1	BIAS	0.2218	0.6542
κ_1	MSE	1.0553	5.9031
κ_2	BIAS	0.0886	0.1020
κ_2	MSE	0.1140	0.1991
α_1	BIAS	0.0018	0.0043
α_1	MSE	4.8067	5.4611
β_1	BIAS	- 0.1441	- 0.2189
β_1	MSE	13.0340	13.6700
α_2	BIAS	- 0.4827	- 1.0851
α_2	MSE	74.9063	91.1043
β_2	BIAS	- 0.0131	- 0.4460
β_2	MSE	132.5285	153.9167
C_1		196	198
C_2		4	2
C_3		0	0

EXAMPLE 9

Table 60 : n=50 , S=50

		MLE	MDE
p	BIAS	0.0007	0.0023
p	MSE	0.0039	0.0043
κ_1	BIAS	1.0865	0.6451
κ_1	MSE	7.3750	8.2768
κ_2	BIAS	0.6039	0.9596
κ_2	MSE	2.6141	6.3622
α_1	BIAS	- 0.7118	- 0.7078
α_1	MSE	14.9707	21.4960
β_1	BIAS	0.3358	0.2468
β_1	MSE	16.7620	18.6348
α_2	BIAS	1.5350	1.4007
α_2	MSE	25.4603	32.6376
β_2	BIAS	- 0.9781	- 1.0590
β_2	MSE	34.7600	31.1453
C_1		50	50
C_2		0	0
C_3		0	0

Table 61 : n=50 , S=200

		MLE	MDE
p	BIAS	0.0004	0.0036
p	MSE	0.0046	0.0048
κ_1	BIAS	0.8819	0.9697
κ_1	MSE	7.2275	13.1476
κ_2	BIAS	0.5471	0.9020
κ_2	MSE	2.2757	4.9175
α_1	BIAS	- 0.8311	- 0.9261
α_1	MSE	18.4390	22.4183
β_1	BIAS	0.2936	0.2178
β_1	MSE	16.5051	17.6111
α_2	BIAS	0.5488	0.6421
α_2	MSE	25.7701	27.1968
β_2	BIAS	- 0.3402	- 0.4005
β_2	MSE	34.3635	34.5688
C_1		200	197
C_2		0	0
C_3		0	3

EXAMPLE 9

Table 62 : n=100 , S=50

		MLE	MDE
p	BIAS	- 0.0024	0.0001
p	MSE	0.0022	0.0022
κ_1	BIAS	0.8531	0.3685
κ_1	MSE	3.6068	4.3174
κ_2	BIAS	0.1809	0.3258
κ_2	MSE	0.7348	1.7564
α_1	BIAS	- 0.3363	- 0.5340
α_1	MSE	10.8274	13.6206
β_1	BIAS	- 0.1632	- 0.1394
β_1	MSE	9.1768	9.8618
α_2	BIAS	0.3676	0.4245
α_2	MSE	12.0229	12.7714
β_2	BIAS	- 0.8096	- 0.7835
β_2	MSE	19.6124	19.3752
C_1		50	50
C_2		0	0
C_3		0	0

Table 63 : n=100 , S=200

		MLE	MDE
p	BIAS	0.0014	0.0023
p	MSE	0.0025	0.0026
κ_1	BIAS	0.5360	0.6341
κ_1	MSE	2.9448	6.4692
κ_2	BIAS	0.2498	0.3917
κ_2	MSE	0.9571	1.9271
α_1	BIAS	- 0.3924	- 0.4954
α_1	MSE	10.3195	11.5501
β_1	BIAS	- 0.0311	- 0.0558
β_1	MSE	9.0520	9.6182
α_2	BIAS	- 0.0686	- 0.0104
α_2	MSE	12.6075	13.1366
β_2	BIAS	- 0.3313	- 0.3611
β_2	MSE	17.7196	17.4556
C_1		200	200
C_2		0	0
C_3		0	0

EXAMPLE 9

Table 64 : n=200 , S=50

		MLE	MDE
p	BIAS	- 0.0072	- 0.0059
p	MSE	0.0014	0.0013
κ_1	BIAS	0.5986	0.3980
κ_1	MSE	1.3364	2.2025
κ_2	BIAS	- 0.0097	0.0922
κ_2	MSE	0.2895	0.7637
α_1	BIAS	- 0.3559	- 0.3916
α_1	MSE	5.5321	7.9131
β_1	BIAS	- 0.1843	- 0.1015
β_1	MSE	4.7782	4.4116
α_2	BIAS	0.3361	0.3157
α_2	MSE	7.0299	7.3817
β_2	BIAS	- 0.5246	- 0.4142
β_2	MSE	7.3334	7.1664
C_1		50	50
C_2		0	0
C_3		0	0

Table 65 : n=200 , S=200

		MLE	MDE
p	BIAS	- 0.0012	- 0.0004
p	MSE	0.0015	0.0014
κ_1	BIAS	0.3600	0.3886
κ_1	MSE	1.1976	2.2348
κ_2	BIAS	0.1135	0.2126
κ_2	MSE	0.4253	0.8461
α_1	BIAS	- 0.1122	- 0.1999
α_1	MSE	4.8362	5.6438
β_1	BIAS	- 0.1230	- 0.1181
β_1	MSE	4.5788	4.6790
α_2	BIAS	0.1108	0.1841
α_2	MSE	6.9841	7.5712
β_2	BIAS	- 0.0449	0.0628
β_2	MSE	8.2542	8.5614
C_1		200	200
C_2		0	0
C_3		0	0

EXAMPLE 10

Table 66 : n=50 , S=50

		MLE	MDE
p	BIAS	- 0.0600	- 0.1554
p	MSE	0.0251	0.0519
κ_1	BIAS	1.2957	2.0480
κ_1	MSE	17.9386	22.9024
κ_2	BIAS	6.8496	2.3838
κ_2	MSE	142.4425	18.6492
α_1	BIAS	0.3181	0.9004
α_1	MSE	37.9711	57.0442
β_1	BIAS	- 0.4909	- 2.1207
β_1	MSE	58.3005	93.4262
α_2	BIAS	- 10.9724	- 24.2915
α_2	MSE	1447.6201	1753.0167
β_2	BIAS	- 8.4345	- 22.4406
β_2	MSE	2089.1845	3041.4176
C_1		42	31
C_2		4	8
C_3		4	11

Table 67 : n=50 , S=200

		MLE	MDE
p	BIAS	- 0.0430	- 0.1305
p	MSE	0.0152	0.0374
κ_1	BIAS	0.8239	1.5352
κ_1	MSE	6.9519	10.3833
κ_2	BIAS	6.4751	1.6740
κ_2	MSE	164.3696	12.1276
α_1	BIAS	0.0087	- 1.0923
α_1	MSE	34.9856	64.5196
β_1	BIAS	- 0.4390	- 2.3454
β_1	MSE	38.8718	85.3624
α_2	BIAS	- 11.8552	- 21.2703
α_2	MSE	1194.9142	1936.9955
β_2	BIAS	- 2.9540	- 27.0956
β_2	MSE	2029.8988	3143.8862
C_1		151	125
C_2		22	27
C_3		27	48

EXAMPLE 10

Table 68 : n=100 , S=50

		MLE	MDE
p	BIAS	- 0.0280	- 0.1064
p	MSE	0.0091	0.0406
κ_1	BIAS	0.3744	1.6870
κ_1	MSE	1.0318	19.6143
κ_2	BIAS	4.5339	1.5533
κ_2	MSE	163.6741	19.1655
α_1	BIAS	0.0842	- 1.1436
α_1	MSE	19.0670	24.2194
β_1	BIAS	- 0.5327	- 0.5539
β_1	MSE	21.9607	33.7945
α_2	BIAS	- 8.2823	- 14.0602
α_2	MSE	1125.9691	1285.9519
β_2	BIAS	- 5.3422	- 20.8947
β_2	MSE	2145.9724	2858.3081
C_1		48	37
C_2		2	5
C_3		0	8

Table 69 : n=100 , S=200

		MLE	MDE
p	BIAS	- 0.0312	- 0.0735
p	MSE	0.0102	0.0225
κ_1	BIAS	0.3806	0.9864
κ_1	MSE	1.1803	6.9230
κ_2	BIAS	3.3730	1.8011
κ_2	MSE	76.1869	17.0945
α_1	BIAS	- 0.3194	- 0.9201
α_1	MSE	19.2053	24.3055
β_1	BIAS	- 0.2419	- 0.8130
β_1	MSE	19.2110	22.6270
α_2	BIAS	- 7.5189	- 10.8910
α_2	MSE	1032.6417	1214.6674
β_2	BIAS	- 4.8246	- 19.3411
β_2	MSE	2034.6382	2627.6920
C_1		186	150
C_2		6	22
C_3		8	28

EXAMPLE 10

Table 70 : n=200 , S=50

		MLE	MDE
p	BIAS	- 0.0147	- 0.0339
p	MSE	0.0043	0.0093
κ_1	BIAS	0.1763	0.4396
κ_1	MSE	0.3882	1.9930
κ_2	BIAS	1.3160	1.4594
κ_2	MSE	12.1050	8.7101
α_1	BIAS	- 0.1767	- 0.2959
α_1	MSE	11.9299	13.7637
β_1	BIAS	- 0.1790	- 0.3801
β_1	MSE	8.5754	9.2740
α_2	BIAS	- 6.4274	- 8.6688
α_2	MSE	312.3544	520.4719
β_2	BIAS	- 9.7974	- 26.0175
β_2	MSE	1659.7122	2666.5172
C_1		50	39
C_2		0	6
C_3		0	5

Table 71 : n=200 , S=200

		MLE	MDE
p	BIAS	- 0.0095	- 0.0352
p	MSE	0.0033	0.0081
κ_1	BIAS	0.1491	0.4204
κ_1	MSE	0.3678	1.2213
κ_2	BIAS	1.2915	1.4843
κ_2	MSE	9.2421	12.4903
α_1	BIAS	0.1464	- 0.3905
α_1	MSE	8.8929	10.0040
β_1	BIAS	- 0.0580	- 0.5903
β_1	MSE	8.3603	9.3494
α_2	BIAS	- 6.6917	- 8.9135
α_2	MSE	298.7982	439.0650
β_2	BIAS	- 2.5332	- 22.0310
β_2	MSE	1256.4316	2206.2353
C_1		200	157
C_2		0	30
C_3		0	13

7.2 The Real data

Schmidt(1976) obtained a data set which described a magnetic pole position. He showed that this data set did not fit a single Fisher model and appeared to fall into two main groups. Consequently Wood(1982) fitted the bimodal model (5.18) to this data set and obtained the following estimates :

$\kappa = 19.3$, $(\alpha_1,\beta_1) = (39,135.7)$ and $(\alpha_2,\beta_2) = (39.5,170)$.

Wood noted that these estimates were in reasonable agreement with values obtained by Schmidt and McDougall(1977) namely

$(\alpha_1,\beta_1) = (42.3,123.5)$ and $(\alpha_2,\beta_2) = (39.3,174.5)$.

We fitted model (5.14) i.e. the mixture of two Fisher distributions, to this data set using both the maximum likelihood and minimum distance methods of estimation. This will enable us to check whether Wood's assumption of equal proportions and equal concentrations is reasonable.

The MLE estimates are :

$p = 0.3320$, $\kappa_1 = 21.4944$, $\kappa_2 = 36.7102$,$(\alpha_1,\beta_1) = (41.9107,121.5850)$ and $(\alpha_2,\beta_2) = (39.3941,172.4335)$.

The MDE estimates are :

$p = 0.2984$, $\kappa_1 = 31.6269$, $\kappa_2 = 38.7327$,$(\alpha_1,\beta_1) = (42.0409,116.0429)$ and $(\alpha_2,\beta_2) = (40.0135,171.9590)$. Both these sets of results are in reasonable agreement apart from the difference in κ_1 .

In fact our estimates of (α_1,β_1) and (α_2,β_2) are in closer agreement with Schmidt and McDougall's values than are Wood's. Also, our results cast some doubt on Wood's assumption of $p = 0.5$ and $\kappa_1 = \kappa_2$.

Out of interest we used our methods to fit the model with $p = 0.5$ and $\kappa_1 = \kappa_2$.

The MLE estimates are : $\kappa_1 = \kappa_2 = 29.7752$, $(\alpha_1,\beta_1) = (41.1340,124.9365)$ and $(\alpha_2,\beta_2) = (39.6829,174.4250)$.

The MDE estimates are : $\kappa_1 = \kappa_2 = 35.0064$, $(\alpha_1,\beta_1) = (39.3025,130.7430)$ and $(\alpha_2,\beta_2) = (41.5953,180.2494)$.

The above results for (α_1,β_1) and (α_2,β_2) using MLE are in excellent agreement with those obtained by Schmidt and McDougall. The results using MDE are not as close but still compare favourably with Wood's estimates. However, the estimates of κ given by both MLE and MDE differ somewhat and there is a big discrepancy between these and the estimate of κ obtained by Wood.

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